Adversarial Bandits: Theory and Algorithms

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Adversarial (a.k.a. Non-Stochastic) Multi-Armed Bandits

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Stochastic MAB is a special case where $\ell_1, \ldots, \ell_T$ are iid generated

A Closer Look

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  - foundation for all other regret measures
  - for games, implies convergence to equilibrium/optimal social welfare

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- highlight how to control the variance of estimators
- highlight the differences between full-info and bandit

Warm-Up: The Expert Problem

The Expert Problem

Freund-Schapire'97

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Not trivial at all even with full information!

The Classical Algorithm

Littlestone-Warmuth'94, Freund-Schapire'97

At round t, sample $i_t \sim p_t \in \Delta_K$ s.t. (for some learning rate $\eta > 0$)

$$\boxed{p_{t,i} \propto \exp\left(-\eta \sum_{\tau<t} \ell_{\tau,i}\right)}$$

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called by many names: Hedge, Multiplicative Weights Update (MWU), ...

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$e^{-z} \le 1 - z + z^2, \ \forall z \ge 0$ and $\ell_{t,i} \ge 0$

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$$\boxed{\ln(1+z)\le z}$$

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Telescoping and rearranging gives:

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Since $\ell_{t,i}^2 \le 1$, picking the best η gives $\mathrm{Reg} = \mathcal{O}(\sqrt{T \ln K})$ (optimal)

A More Modern View

Hedge is a special case of Follow-the-Regularized-Leader (FTRL):

$$p_t = \operatorname*{argmin}_{p \in \Delta_K} \left\langle p, \sum_{\tau < t} \ell_\tau \right\rangle + \frac{1}{\eta} \psi(p)$$

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stability term

penalty term

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- for Shannon entropy: $\|\ell_t\|_{p_t}^2 = \sum_i p_{t,i}\ell_{t,i}^2$

From Full-Info to Bandit

The Exp3 Algorithm

Auer-CesaBianchi-Freund-Schapire'02

Obvious issue in MAB: only one coordinate of ℓ_t is observed

The Exp3 Algorithm

Auer-CesaBianchi-Freund-Schapire'02

Obvious issue in MAB: only one coordinate of ℓ_t is observed

Solution: construct an importance-weighted estimator $\widehat{\ell}_t$ with

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Key challenge: the **variance** of the estimator can be huge

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Note the gap between this and Exp3's regret bound $\mathcal{O}(\sqrt{TK \ln K})$

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Audibert-Bubeck’09, Abernethy-Lee-Tewari’15

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Magical bonus: not only minimax optimal for adversarial losses, but (surprisingly) also *instance-optimal for stochastic losses!* (Zimmert-Seldin'19)

Beyond Minimax Optimality:
Adaptive and Problem-Dependent Regret Bounds

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variance of losses $Q = \frac{1}{K}\sum_{t,i}(\ell_{t,i} - \frac{1}{T}L_i)$	$\widetilde{\mathcal{O}}(\sqrt{QK})$	Hazan-Kale'11, Bubeck-Cohen-Li'17

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Robustness versus Adaptivity

Worst-case robustness ($\sqrt{TK}$-regret) might be overly pessimistic. Can we adapt to easier instances with smaller regret?

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imply **faster convergence** in games

Achieving Small-Loss Bounds

Foster-Li-Lykouris-Sridharan-Tardos'16

Consider FTRL with the log-barrier regularizer $\psi(p) = -\sum_{i=1}^{K} \ln p_i$,

$$p_t = \operatorname*{argmin}_{p \in \Delta_K} \left\langle p, \sum_{\tau < t} \widehat{\ell}_\tau \right\rangle + \frac{1}{\eta}\psi(p)$$

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	Shannon	**$1/2$-Tsallis**	**log-barrier**
penalty	$\ln K$	$\sqrt{K}$	$K \ln T$
stability	K	$\sqrt{K}$	1

Recal ... $_1 \|\widehat{\ell}_t\|_{p_t}^2$

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Wei-Luo'18

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  - log-barrier + Shannon entropy (Bubeck-Cohen-Li'18, Bubeck-Li-Luo-Wei'19, Lee-Luo-Zhang'20, Ito-Tsuchiya-Honda'22)
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