Lecture 5. Online Convex Optimization

Advanced Optimization (Fall 2024)

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Outline

- Online Learning
- Online Convex Optimization
- Connection to Offline Learning

Part 1. Online Learning

- Statistical Learning

- Online Learning: Problem and Measure

- Online Convex Optimization

A Brief Review of Statistical Learning

- The fundamental goal of (supervised) learning: ***Risk Minimization***

$$\min_{h \in \mathcal{H}} \mathbb{E}_{(\mathbf{x},y)\sim\mathcal{D}}[\ell(h(\mathbf{x}), y)],$$

where

- h denotes the hypothesis (model) from the hypothesis space $\mathcal{H}$.
- $(\mathbf{x}, y)$ is an instance chosen from an unknown distribution $\mathcal{D}$.
- $\ell(h(\mathbf{x}), y)$ denotes the loss of using hypothesis h on the instance $(\mathbf{x}, y)$.

A Brief Review of Statistical Learning

- Given a data distribution $\mathcal{D}$, a predictive model $h : \mathcal{X} \mapsto \hat{\mathcal{Y}}$, and the loss function $\ell : \hat{\mathcal{Y}} \times \mathcal{Y} \mapsto \mathbb{R}$, the ***expected risk*** is defined by

$$R(h) = \mathbb{E}_{(\mathbf{x},y)\sim\mathcal{D}}[\ell(h(\mathbf{x}), y)].$$

- In practice, we can only access to samples $S = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)\}$. Thus, the following *empirical risk* is naturally defined:

$$\widehat{R}_S(h) = \frac{1}{m}\sum_{i=1}^{m} \ell(h(\mathbf{x}_i), y_i).$$

A Brief Review of Statistical Learning

- A successful paradigm : characterization of sample complexity

  - excess risk bound

$$R(\hat{h}) - \inf_{h \in \mathcal{H}} R(h) \leq \mathcal{O}\left(\frac{1}{\sqrt{m}}\right).$$

  - generalization error bound

$$\hat{R}_S(\hat{h}) - R(\hat{h}) \leq \mathcal{O}\left(\frac{1}{\sqrt{m}}\right).$$

Offline Towards Online Learning

- Traditional statistical machine learning
  - The training data are available ***offline***
  - Learning model is trained based on the offline data in a ***batch*** setting

- Online learning scenario
  - In real applications, data are in the form of ***stream***
  - New data are being collected all the time: after observing a new data point, the model should be ***online updated*** at a constant cost

A Formulation of Online Learning

- We model online learning from the lens of ***optimization***.
- Online learning is formulated as a ***repeated game*** between
  - ❑ ***Player***: essentially the learner, or you can think as the "learning model"
  - ❑ ***Environments***: an abstraction of all factors evaluating the model.

At each round $t = 1, 2, \cdots$

(1) the player first picks a model $\mathbf{w}_t \in \mathcal{W}$;

(2) and simultaneously environments pick an online function $f_t : \mathcal{W} \to \mathbb{R}$;

(3) the player suffers loss $f_t(\mathbf{w}_t)$, observes some information about f_t and updates the model.

Online Learning: Formulation

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- An example of online function $f_t : \mathcal{W} \mapsto \mathbb{R}$.

Considering the task of ***online classification***, we have

(i) the loss $\ell : \hat{\mathcal{Y}} \times \mathcal{Y} \mapsto \mathbb{R}$, and

(ii) the hypothesis function $h : \mathcal{W} \times \mathcal{X} \mapsto \hat{\mathcal{Y}}$.

$$\Longrightarrow \quad f_t(\mathbf{w}) = \ell(h(\mathbf{w}; \mathbf{x}_t), y_t)$$

$$= \ell(\mathbf{w}^\top \mathbf{x}_t, y_t) \quad \textit{for simplicity}$$

Online Learning: Formulation

At each round $t = 1, 2, \cdots$

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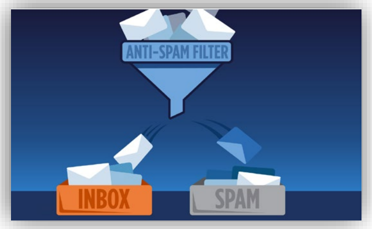
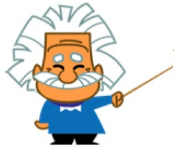
Spam filtering





(1) Player submits a spam classifier $\mathbf{w}_t$

⇓

(2) A mail is revealed whether it is a spam

⇓

(3) Player suffers loss $f_t(\mathbf{w}_t)$ and updates model

Performance Measure

- Recall in the statistical learning:

Risk $$R(h) \triangleq \mathbb{E}_{(\mathbf{x},y)\sim\mathcal{D}}\big[\ell(h(\mathbf{x}),y)\big].$$

- In online learning:

Sequential Risk $$\hat{R}(\{\mathbf{w}_t\}_{t=1}^T) \triangleq \sum_{t=1}^T f_t(\mathbf{w}_t) = \sum_{t=1}^T \ell(\mathbf{w}_t^\top \mathbf{x}_t, y_t).$$

meaning: cumulative loss of online models trained on the growing data stream $S_t = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_t, y_t)\}$.

Performance Measure

- In offline learning, we use ***excess risk*** as measure for $\hat{h}$:

$$R(\hat{h}) - \min_{h \in \mathcal{H}} R(h)$$

$$= \mathbb{E}_{(\mathbf{x},y)\sim\mathcal{D}}[\ell(\mathbf{w}^\top \mathbf{x}, y)] - \min_{\mathbf{w}\in\mathcal{W}} \mathbb{E}_{(\mathbf{x},y)\sim\mathcal{D}}[\ell(\mathbf{w}^\top \mathbf{x}, y)]$$

simply using a linear model $\mathbf{w}$ *to parametrize the hypothesis h*

- In online learning, we define ***regret*** as measure for sequence $\{\mathbf{w}_t\}_{t=1}^T$:

$$R(\{\mathbf{w}_t\}_{t=1}^T) - \min_{\mathbf{w}\in\mathcal{W}} R(\mathbf{w})$$

$$= \sum_{t=1}^T \ell(\mathbf{w}_t^\top \mathbf{x}_t, y_t) - \min_{\mathbf{w}\in\mathcal{W}} \sum_{t=1}^T \ell(\mathbf{w}^\top \mathbf{x}_t, y_t)$$

$$\text{Regret}_T = \sum_{t=1}^T f_t(\mathbf{w}_t) - \min_{\mathbf{w}\in\mathcal{W}} \sum_{t=1}^T f_t(\mathbf{w})$$

benchmark performance with the offline model (optimal in hindsight)

Regret Measure

- We use regret to measure the online learning algorithm

$$\text{Regret}_T = \sum_{t=1}^{T} f_t(\mathbf{w}_t) - \min_{\mathbf{w}\in\mathcal{W}} \sum_{t=1}^{T} f_t(\mathbf{w})$$

benchmark performance with the offline model (optimal in hindsight)

- We hope the regret be sub-linear dependence with T.

$$\frac{\text{Regret}_T}{T} \to 0 \text{ as } T \to \infty$$

Hannan Consistency

ALT'16

Hannan Consistency in On-Line Learning in Case of Unbounded Losses Under Partial Monitoring*,**

Chamy Allenberg[1], Peter Auer[2], László Györfi[3], and György Ottucsák[3]

Is Online Learning (provably) solvable?

- In general, the online learning formulation is ***hard*** to solve.

> At each round $t = 1, 2, \cdots$
>
> (1) the player first picks a model $\mathbf{w}_t \in \mathcal{W}$;
>
> (2) and simultaneously environments pick an online function $f_t : \mathcal{W} \to \mathbb{R}$;
>
> (3) the player suffers loss $f_t(\mathbf{w}_t)$, observes some information about f_t and updates the model.

⇨ A Trackable Case: ***Online Convex Optimization***

requiring feasible domain and online functions to be convex

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⇨ A Trackable Case: ***Online Convex Optimization***

requiring feasible domain and online functions to be convex

Online Convex Optimization

- Requirements:
  (1) feasible domain is a convex set
  (2) online functions are convex

At each round $t = 1, 2, \cdots$

(1) the player first picks a model $\mathbf{x}_t$ from a convex set $\mathcal{X} \subseteq \mathbb{R}^d$;

(2) and environments pick an online convex function $f_t : \mathcal{X} \to \mathbb{R}$;

(3) the player suffers loss $f_t(\mathbf{x}_t)$, observes some information about f_t and updates the model.

Henceforth, we use x (and $\mathcal{X}$) instead of w (and $\mathcal{W}$) for consistency with opt. language.

OCO: Different Feedback

At each round $t = 1, 2, \cdots$

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on the feedback information:

- *full information*: observe entire f_t (or at least gradient $\nabla f_t(\mathbf{x}_t)$)

- *partial information (bandits)*: observe function value $f_t(\mathbf{x}_t)$ only

less information

full information

horse racing

partial information

multi-armed bandits

OCO: Different Environments

At each round $t = 1, 2, \cdots$

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(3) the player suffers loss $f_t(\mathbf{x}_t)$, observes some information about f_t and updates the model.

on the difficulty of environments:

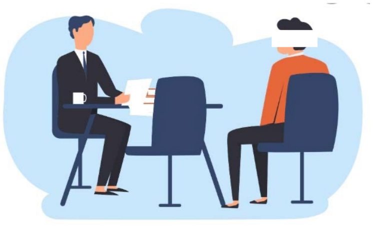
- stochastic setting
- adversarial setting { oblivious / adaptive (non-oblivious)

less restricted but harder

oblivious adversary

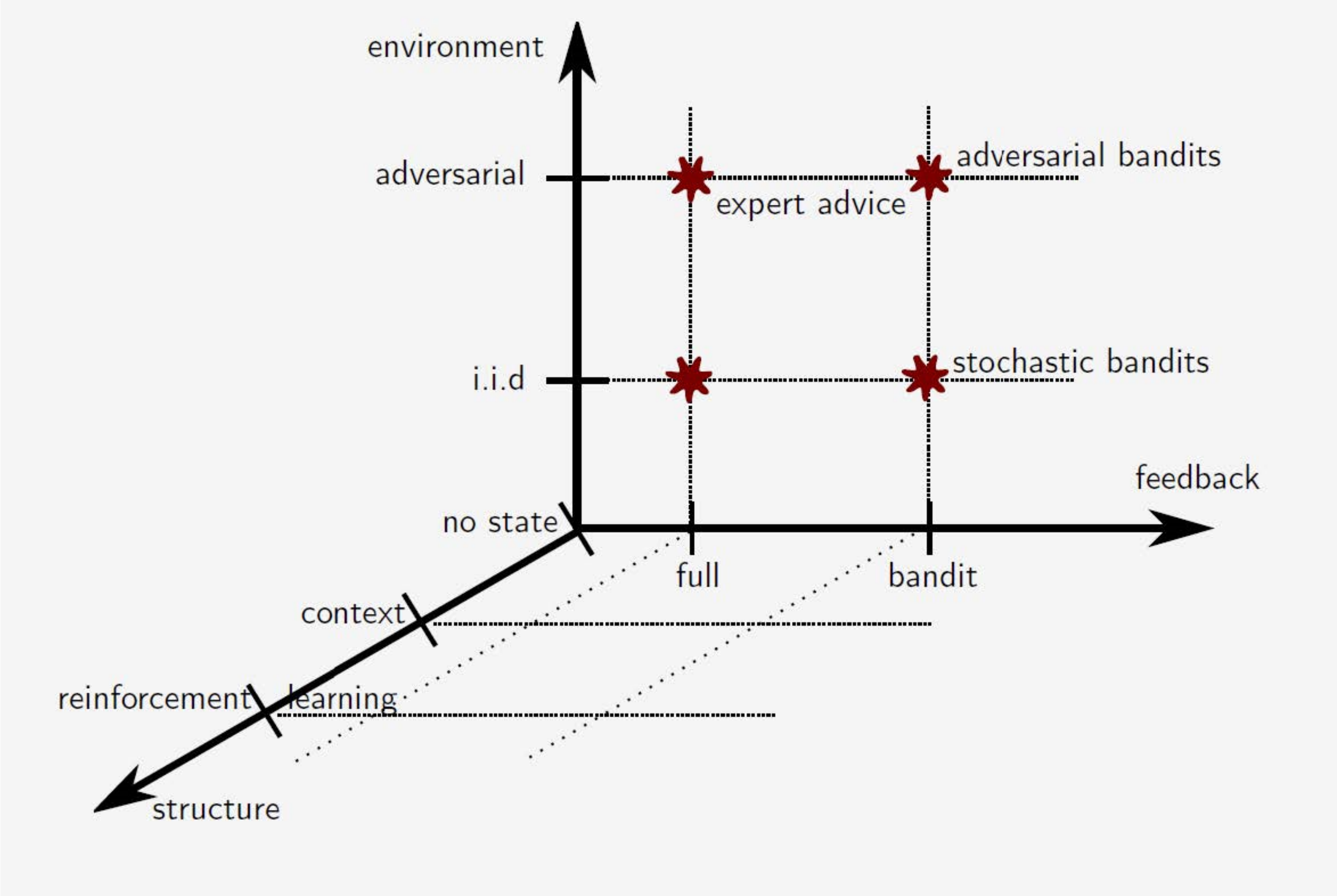
examination

adaptive adversary



interview

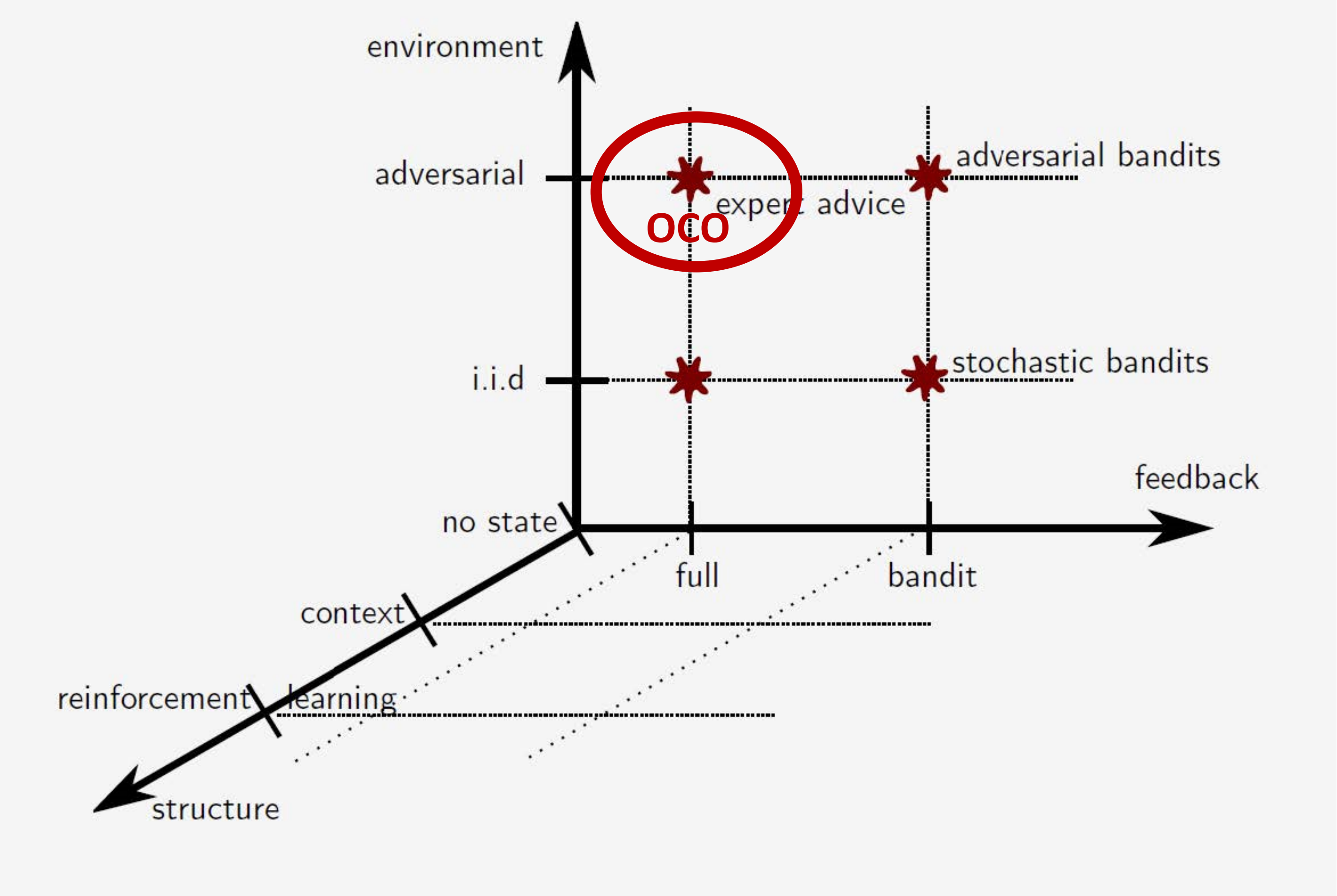
The Space of Online Learning Problems



- Full-information setting:
  - Online Convex Optimization
  - Prediction with Expert Advice
  - …
- Partial-information setting:
  - Multi-Armed Bandits
  - Linear Bandits
  - Parametric Bandits
  - Bandit Convex Optimization
  - …

Yevgeny Seldin. The Space of Online Learning Problems, ECML-PKDD, Porto, Portugal, 2015.

The Space of Online Learning Problems



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Part 2. Online Convex Optimization

- Convex Functions

- Strongly Convex Functions

- Exponentially Concave Functions

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OCO: Convex Functions

Definition 2 (Convex Function)**.** A function $f : \mathcal{X} \mapsto \mathbb{R}$ is convex if for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$

$$\forall \alpha \in [0,1], f((1-\alpha)\mathbf{x} + \alpha \mathbf{y}) \leq (1-\alpha) f(\mathbf{x}) + \alpha f(\mathbf{y}).$$

Equivalently, if f is differentiable, we have that $\forall \mathbf{x}, \mathbf{y} \in \mathcal{X}$,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}).$$

OCO: OGD Algorithm

Online Gradient Descent (OGD)

At each round $t = 1, 2, \cdots$

1. the player first picks a model $\mathbf{x}_t \in \mathcal{X}$;
2. and simultaneously environments pick a ***convex*** online function $f_t : \mathcal{X} \to \mathbb{R}$;
3. the player suffers loss $f_t(\mathbf{x}_t)$, observes the information of f_t and update the model according to $\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}\left[\mathbf{x}_t - \eta_t \nabla f_t(\mathbf{x}_t)\right]$.

- $\Pi_{\mathcal{X}}[\mathbf{y}] = \arg\min_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x} - \mathbf{y}\|_2$ denotes the Euclidean projection onto the feasible set $\mathcal{X}$.
- This belongs to the full-information setting, so player can access the gradient $\nabla f_t(\mathbf{x}_t)$.

Actually, only gradient is required, so it's also called ***gradient-feedback*** OCO model.

Regret Analysis of OGD

- The following assumptions are required for standard analysis.

Assumption 1 (Convexity). The feasible set $\mathcal{X}$ is closed and convex in Euclidean space, and $f_1, \ldots, f_T$ are convex functions.

Assumption 2 (Bounded Domain). The diameter of the feasible domain $\mathcal{X}$ is upper bounded by D, i.e., $\forall \mathbf{x}, \mathbf{y} \in \mathcal{X}, \|\mathbf{x} - \mathbf{y}\| \leq D$.

Assumption 3 (Bounded Gradient). The norm of the subgradients is upper bounded by G, i.e., $\|\nabla f_t(\mathbf{x})\| \leq G$ for all $\mathbf{x} \in \mathcal{X}$ and $t \in [T]$.

Regret Analysis of OGD

Theorem 3 (Regret bound for OGD). *Under Assumptions 1, 2 and 3, online gradient descent (OGD) with step sizes $\eta_t = \frac{D}{G\sqrt{t}}$ for $t \in [T]$ guarantees:*

$$\text{Regret}_T = \sum_{t=1}^{T} f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^{T} f_t(\mathbf{x}) \leq \frac{3}{2} GD\sqrt{T} = \mathcal{O}(\sqrt{T}).$$

The First Gradient Descent Lemma

Lemma 1. *Suppose that f is proper, closed and convex; the feasible domain $\mathcal{X}$ is nonempty, closed and convex. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by the gradient descent method. Then for any $\mathbf{u} \in \mathcal{X}^\star$ and $t \geq 0$,*

$$\|\mathbf{x}_{t+1} - \mathbf{u}\|^2 \leq \|\mathbf{x}_t - \mathbf{u}\|^2 - 2\eta_t(f_t(\mathbf{x}_t) - f_t(\mathbf{u})) + \eta_t^2\|\nabla f_t(\mathbf{x}_t)\|^2.$$

Proof:

$$\begin{aligned}
\|\mathbf{x}_{t+1} - \mathbf{u}\|^2 &= \|\Pi_{\mathcal{X}}[\mathbf{x}_t - \eta_t \nabla f_t(\mathbf{x}_t)] - \mathbf{u}\|^2 \quad \text{(GD)} \\
&\leq \|\mathbf{x}_t - \eta_t \nabla f_t(\mathbf{x}_t) - \mathbf{u}\|^2 \quad \text{(Pythagoras Theorem)} \\
&= \|\mathbf{x}_t - \mathbf{u}\|^2 - 2\eta_t \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{u}\rangle + \eta_t^2 \|\nabla f_t(\mathbf{x}_t)\|^2 \\
&\leq \|\mathbf{x}_t - \mathbf{u}\|^2 - 2\eta_t(f_t(\mathbf{x}_t) - f_t(\mathbf{u})) + \eta_t^2 \|\nabla f_t(\mathbf{x}_t)\|^2
\end{aligned}$$

(convexity: $f_t(\mathbf{x}_t) - f_t(\mathbf{u}) = f_t(\mathbf{x}_t) - f_t(\mathbf{u}) \leq \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{u}\rangle$) □

Proof for OGD Regret Bound

Proof: We use the first gradient descent lemma to analyze online gradient descent.

Lemma 1. *Suppose that f is proper, closed and convex; the feasible domain $\mathcal{X}$ is nonempty, closed and convex. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by the gradient descent method. Then for any $\mathbf{u} \in \mathcal{X}^\star$ and $t \geq 0$,*

$$\|\mathbf{x}_{t+1} - \mathbf{u}\|^2 \leq \|\mathbf{x}_t - \mathbf{u}\|^2 - 2\eta_t(f(\mathbf{x}_t) - f(\mathbf{u})) + \eta_t^2\|\nabla f(\mathbf{x}_t)\|^2.$$

By Lemma 1 and the gradient boundedness, we have

$$2(f_t(\mathbf{x}_t) - f_t(\mathbf{u})) \leq \frac{\|\mathbf{x}_t - \mathbf{u}\|^2 - \|\mathbf{x}_{t+1} - \mathbf{u}\|^2}{\eta_t} + \eta_t G^2$$

Proof for OGD Regret Bound

Proof: By setting $\eta_t = \frac{D}{G\sqrt{t}}$ $\left(\text{with } \frac{1}{\eta_0} := 0\right)$, summing over T:

$$
\begin{aligned}
2\left(\sum_{t=1}^{T} f_t(\mathbf{x}_t) - f_t(\mathbf{u})\right) &\le \sum_{t=1}^{T} \frac{\|\mathbf{x}_t - \mathbf{u}\|^2 - \|\mathbf{x}_{t+1} - \mathbf{u}\|^2}{\eta_t} + G^2 \sum_{t=1}^{T} \eta_t \qquad \text{(GD lemma)} \\
&\le \sum_{t=1}^{T} \|\mathbf{x}_t - \mathbf{u}\|^2 \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right) + G^2 \sum_{t=1}^{T} \eta_t \qquad (\|\mathbf{x}_{T+1} - \mathbf{u}\|^2 \ge 0) \\
&\le D^2 \sum_{t=1}^{T} \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right) + G^2 \sum_{t=1}^{T} \eta_t \\
&\le D^2 \frac{1}{\eta_T} + G^2 \sum_{t=1}^{T} \eta_t \qquad \left(\eta_t = \frac{D}{G\sqrt{t}} \text{ and } \textstyle\sum_{t=1}^{T} \frac{1}{\sqrt{t}} \le 2\sqrt{T}\right) \\
&\le 3DG\sqrt{T}.
\end{aligned}
$$

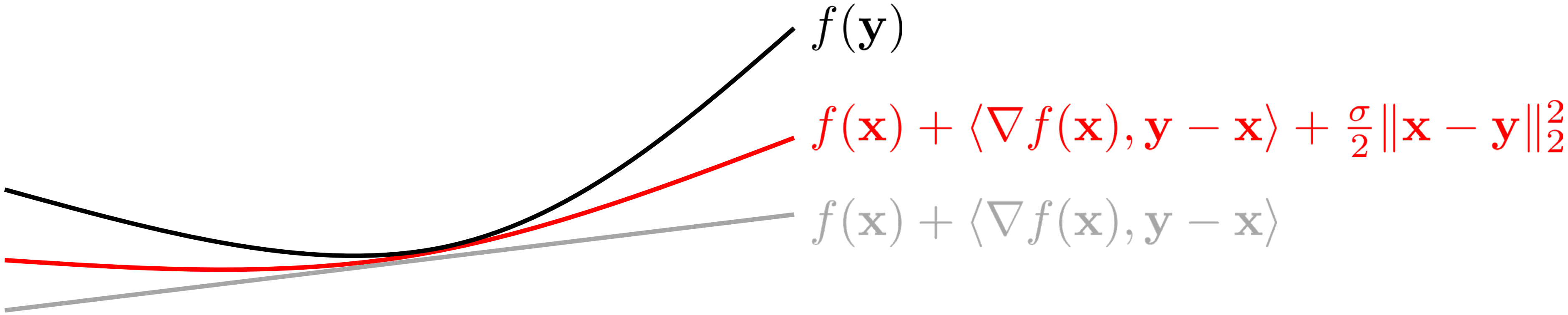
□

OCO: Strongly Convex Functions

Definition 3 (Strong Convexity). A function f is σ-strongly convex if, for any $\mathbf{x}, \mathbf{y} \in \operatorname{dom} f$,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) + \frac{\sigma}{2}\|\mathbf{y} - \mathbf{x}\|^2,$$

or equivalently, $\nabla^2 f(\mathbf{x}) \succeq \alpha I$.



OGD for Strongly Convex Functions

Online Gradient Descent (OGD)

At each round $t = 1, 2, \cdots$

(1) the player first picks a model $\mathbf{x}_t \in \mathcal{X}$;

(2) and simultaneously environments pick a ***strongly convex function*** $f_t : \mathcal{X} \to \mathbb{R}$;

(3) the player suffers loss $f_t(\mathbf{x}_t)$, observes the information of f_t and update the model according to $\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}\left[\mathbf{x}_t - \eta_t \nabla f_t(\mathbf{x}_t)\right]$.

OGD for Strongly Convex Loss

Theorem 4 (Regret bound for strongly-convex functions). *Under Assumption 1 and Assumption 3, for σ-strongly convex loss functions, online gradient descent with step sizes $\eta_t = \frac{1}{\sigma t}$ achieves the following guarantee*

$$\text{Regret}_T \leq \frac{G^2}{2\sigma}(1 + \log T) = \mathcal{O}(\log T).$$

- Strongly convex case compared with convex case: $\mathcal{O}(\log T)$ vs. $\mathcal{O}(\sqrt{T})$
- A caveat is that we now don't need Assumption 2 (bounded domain).

OCO with Strongly Convex Functions

Proof: we start by extending *the first GD lemma* to strongly convex case.

Strongly convex case:

$$
\begin{aligned}
\|\mathbf{x}_{t+1}-\mathbf{u}\|^2 &\le \|\mathbf{x}_t-\mathbf{u}\|^2 - 2\eta_t\langle\nabla f_t(\mathbf{x}_t),\mathbf{x}_t-\mathbf{u}\rangle + \eta_t^2\|\nabla f_t(\mathbf{x}_t)\|^2 \\
&\le \|\mathbf{x}_t-\mathbf{u}\|^2 - 2\eta_t\left(f_t(\mathbf{x}_t)-f_t(\mathbf{u})+\frac{\sigma}{2}\|\mathbf{x}_t-\mathbf{u}\|^2\right) + \eta_t^2\|\nabla f_t(\mathbf{x}_t)\|^2
\end{aligned}
$$

(strong convexity: $f_t(\mathbf{x}_t)-f_t(\mathbf{u})+\frac{\sigma}{2}\|\mathbf{x}_t-\mathbf{u}\|^2 \le \langle\nabla f_t(\mathbf{x}_t),\mathbf{x}_t-\mathbf{u}\rangle$)

$$
\le (1-\sigma\eta_t)\|\mathbf{x}_t-\mathbf{u}\|^2 - 2\eta_t\left(f_t(\mathbf{x}_t)-f_t(\mathbf{u})\right) + \eta_t^2\|\nabla f_t(\mathbf{x}_t)\|^2
$$

$$
\Longrightarrow f_t(\mathbf{x}_t)-f_t(\mathbf{u}) \le \frac{\eta_t^{-1}-\sigma}{2}\|\mathbf{x}_t-\mathbf{u}\|^2 - \frac{\eta_t^{-1}}{2}\|\mathbf{x}_{t+1}-\mathbf{u}\|^2 + \frac{\eta_t G^2}{2}
$$

(rearranging)

OCO with Strongly Convex Functions

Proof: $$f_t(\mathbf{x}_t) - f_t(\mathbf{u}) \leq \frac{\eta_t^{-1} - \sigma}{2}\|\mathbf{x}_t - \mathbf{u}\|^2 - \frac{\eta_t^{-1}}{2}\|\mathbf{x}_{t+1} - \mathbf{u}\|^2 + \frac{\eta_t G^2}{2}$$

Summing from $t = 1$ to T, setting $\eta_t = \frac{1}{\sigma t}$ $\left(\text{define } \frac{1}{\eta_0} := 0\right)$:

$$
\begin{aligned}
2\sum_{t=1}^{T}\left(f_t(\mathbf{x}_t) - f_t(\mathbf{u})\right) &\leq \sum_{t=1}^{T}\|\mathbf{x}_t - \mathbf{u}\|^2\left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} - \sigma\right) + G^2\sum_{t=1}^{T}\eta_t \quad \left(\frac{1}{\eta_0} := 0\right) \\
&= 0 + G^2\sum_{t=1}^{T}\frac{1}{\sigma t} \quad \left(\frac{1}{\eta_0} \triangleq 0, \|\mathbf{x}_{T+1} - \mathbf{u}\|^2 \geq 0, \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} - \sigma = 0\right) \\
&\leq \frac{G^2}{\sigma}(1 + \log T).
\end{aligned}
$$

□

Comparison of (Strongly) Convex Problems

Convex Problem	***Strongly Convex Problem***
Property: $f_t(\mathbf{x}) \geq f_t(\mathbf{y}) + \nabla f_t(\mathbf{y})^\top (\mathbf{x} - \mathbf{y})$	Property: $f_t(\mathbf{y}) \geq f_t(\mathbf{x}) + \nabla f_t(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) + \frac{\sigma}{2}\|\mathbf{y} - \mathbf{x}\|^2$
OGD: $\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}\left[\mathbf{x}_t - \frac{1}{\sqrt{t}}\nabla f_t(\mathbf{x}_t)\right]$	OGD: $\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}\left[\mathbf{x}_t - \frac{1}{\sigma t}\nabla f_t(\mathbf{x}_t)\right]$
$\text{Regret}_T \leq \frac{3}{2} GD\sqrt{T}$	$\text{Regret}_T \leq \frac{G^2}{2\sigma}(1 + \log T)$

Can we explore broader function classes with a regret rate faster than $\sqrt{T}$?

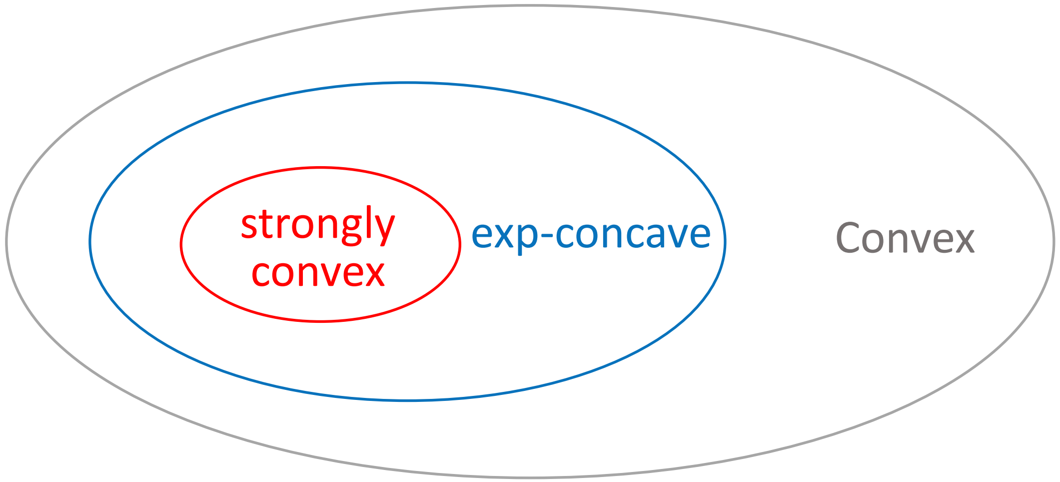
OCO: Exponentially-concave Functions

Definition 2 (Exp-concavity). A convex function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is defined to be α-exp-concave over $\mathcal{X} \subseteq \mathbb{R}^d$ if the function g is concave, where $g : \mathcal{X} \mapsto \mathbb{R}$ is defined as

$$g(\mathbf{x}) = e^{-\alpha f(\mathbf{x})}.$$

Directly employ OGD: Regret bound $\mathcal{O}(\sqrt{T})$.

But actually we can get a **tighter** bound!



An Example for Exp-concave Learning

- Universal Portfolio Selection 
  - a total of d stocks in the stock market.
  - each round, the player chooses stocks by a distribution $\mathbf{x}_t \in \Delta_d$.
  - the market returns the price ratio $\boldsymbol{\theta}_t$ between iter t and $t+1$,

$$\boldsymbol{\theta}_t(i) = \frac{\text{price of stock}_i \text{ at time } t+1}{\text{price of stock}_i \text{ at time } t}$$

which means that our final wealth W_T will be: $W_T = W_1 \cdot \prod_{t=1}^{T} \boldsymbol{\theta}_t^\top \mathbf{x}_t$

$\Longrightarrow$ Our goal is to maximize our wealth at time T.

An Example for Exp-concave Learning

- Universal Portfolio Selection 

  - we hope to maximize the logarithm of W_T

$$\log \frac{W_T}{W_1} = \sum_{t=1}^{T} \log \boldsymbol{\theta}_t^\top \mathbf{x}_t$$

  - using OCO framework,

$$f_t(\mathbf{x}) = \log(\boldsymbol{\theta}_t^\top \mathbf{x})$$

At each round $t = 1, 2, \cdots$

(1) the player first picks a model $\mathbf{x}_t \in \Delta_d$;

(2) and simultaneously environments pick an online function $f_t : \mathcal{X} \to \mathbb{R}$;

(3) the player get a ***gain*** $f_t(\mathbf{x}_t) = \log(\boldsymbol{\theta}_t^\top \mathbf{x}_t)$, observes f_t and updates the model.

- Goal: $\text{Regret}_T = \max_{\mathbf{x}^\star \in \Delta_d} \sum_{t=1}^{T} f_t(\mathbf{x}^\star) - \sum_{t=1}^{T} f_t(\mathbf{x}_t)$

online function is exp-concave

Exponential-concave Function

Lemma 3 (Property of Exp-concavity). *Let $f : \mathcal{X} \to \mathbb{R}$ be an α-exp-concave function, and D, G denote the diameter of $\mathcal{X}$ and a bound on the (sub)gradients of f respectively. The following holds for all $\gamma \leq \frac{1}{2}\min\left\{\frac{1}{GD}, \alpha\right\}$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$:*

$$f(\mathbf{x}) \geq f(\mathbf{y}) + \nabla f(\mathbf{y})^\top(\mathbf{x} - \mathbf{y}) + \frac{\gamma}{2}(\mathbf{x} - \mathbf{y})^\top \nabla f(\mathbf{y}) \nabla f(\mathbf{y})^\top (\mathbf{x} - \mathbf{y}).$$

Proof. Recall that f is α-exp-concave if and only if $e^{-\alpha f(\mathbf{x})}$ is concave.

As $2\gamma \leq \alpha$, $e^{-2\gamma f(\mathbf{x})} = \left(e^{-\alpha f(\mathbf{x})}\right)^{2\gamma/\alpha}$ is also concave and thus is 2γ-exp-concave.

$$e^{-2\gamma f(\mathbf{x})} - e^{-2\gamma f(\mathbf{y})} \leq \left\langle \mathbf{x} - \mathbf{y}, -2\gamma e^{-2\gamma f(\mathbf{y})} \nabla f(\mathbf{y}) \right\rangle.$$

(concavity)

Exponential-concave Function

Lemma 3 (Property of Exp-concavity). *Let $f : \mathcal{X} \to \mathbb{R}$ be an α-exp-concave function, and D, G denote the diameter of $\mathcal{X}$ and a bound on the (sub)gradients of f respectively. The following holds for all $\gamma \leq \frac{1}{2}\min\left\{\frac{1}{GD}, \alpha\right\}$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$:*

$$f(\mathbf{x}) \geq f(\mathbf{y}) + \nabla f(\mathbf{y})^\top(\mathbf{x}-\mathbf{y}) + \frac{\gamma}{2}(\mathbf{x}-\mathbf{y})^\top \nabla f(\mathbf{y}) \nabla f(\mathbf{y})^\top (\mathbf{x}-\mathbf{y}).$$

Proof. Dividing $e^{-2\gamma f(\mathbf{y})}$ at both sides achieves

$$f(\mathbf{y}) - f(\mathbf{x}) \leq \frac{1}{2\gamma}\log\left(1 + 2\gamma\langle \mathbf{y}-\mathbf{x}, \nabla f(\mathbf{y})\rangle\right).$$

Our constructive condition $\gamma \leq \frac{1}{2}\min\left\{\frac{1}{GD}, \alpha\right\}$ ensures $|2\gamma\langle \mathbf{y}-\mathbf{x}, \nabla f(\mathbf{y})\rangle| \leq 1$,

$$f(\mathbf{y}) - f(\mathbf{x}) \leq \langle \mathbf{y}-\mathbf{x}, \nabla f(\mathbf{y})\rangle - \frac{\gamma}{2}\langle \mathbf{y}-\mathbf{x}, \nabla f(\mathbf{y})\rangle^2$$

$\left(\log(1+x) \leq x - \frac{1}{4}x^2\right)$ holds for $(|x| \leq 1)$ □

A Comparison of Different Curvatures

- Convex

$$f(\mathbf{x}) \geq f(\mathbf{y}) + \nabla f(\mathbf{y})^\top (\mathbf{x} - \mathbf{y})$$

- Strongly Convex

$$f(\mathbf{x}) \geq f(\mathbf{y}) + \nabla f(\mathbf{y})^\top (\mathbf{x} - \mathbf{y}) + \frac{\sigma}{2} \|\mathbf{x} - \mathbf{y}\|_2^2$$

- Exponentially Concave

$$\begin{aligned} f(\mathbf{x}) &\geq f(\mathbf{y}) + \nabla f(\mathbf{y})^\top (\mathbf{x} - \mathbf{y}) + \frac{\gamma}{2} (\mathbf{x} - \mathbf{y})^\top \nabla f(\mathbf{y}) \nabla f(\mathbf{y})^\top (\mathbf{x} - \mathbf{y}) \\ &= f(\mathbf{y}) + \nabla f(\mathbf{y})^\top (\mathbf{x} - \mathbf{y}) + \frac{\gamma}{2} \|\mathbf{x} - \mathbf{y}\|^2_{\nabla f(\mathbf{y}) \nabla f(\mathbf{y})^\top} \end{aligned}$$

Exponential-concave Function

Lemma 3 (Property of Exp-concavity). *Let $f : \mathcal{X} \to \mathbb{R}$ be an α-exp-concave function, and D, G denote the diameter of $\mathcal{X}$ and a bound on the (sub)gradients of f respectively. The following holds for all $\gamma \leq \frac{1}{2}\min\left\{\frac{1}{GD}, \alpha\right\}$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$:*

$$\begin{aligned} f(\mathbf{x}) &\geq f(\mathbf{y}) + \nabla f(\mathbf{y})^\top (\mathbf{x}-\mathbf{y}) + \frac{\gamma}{2}(\mathbf{x}-\mathbf{y})^\top \nabla f(\mathbf{y}) \nabla f(\mathbf{y})^\top (\mathbf{x}-\mathbf{y}) \\ &= f(\mathbf{y}) + \nabla f(\mathbf{y})^\top (\mathbf{x}-\mathbf{y}) + \frac{\gamma}{2}\|\mathbf{x}-\mathbf{y}\|^2_{\nabla f(\mathbf{y}) \nabla f(\mathbf{y})^\top} \end{aligned}$$

Algorithmic intuition:

- For convex loss, we use 2-norm to encode the structure of the space.
- Can we exploit ***local structures*** of exp-concave loss to improve the regret?

Intuition

- Convex $$f_t(\mathbf{x}) \geq f_t(\mathbf{x}_t) + \nabla f_t(\mathbf{x}_t)^\top (\mathbf{x} - \mathbf{x}_t)$$

$$\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}\left[\mathbf{x}_t - \eta_t \nabla f_t(\mathbf{x}_t)\right]$$ OGD with $\eta_t = \mathcal{O}(1/\sqrt{t})$

- Strongly convex $$f_t(\mathbf{x}) \geq f_t(\mathbf{x}_t) + \nabla f_t(\mathbf{x}_t)^\top (\mathbf{x} - \mathbf{x}_t) + \frac{\sigma}{2}\|\mathbf{x} - \mathbf{x}_t\|_2^2$$

$$\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}\left[\mathbf{x}_t - \eta_t \nabla f_t(\mathbf{x}_t)\right]$$ OGD with $\eta_t = \mathcal{O}(1/t)$

- Exp-concave $$f_t(\mathbf{x}) \geq f_t(\mathbf{x}_t) + \nabla f_t(\mathbf{x}_t)^\top (\mathbf{x} - \mathbf{x}_t) + \frac{\gamma}{2}\|\mathbf{x} - \mathbf{x}_t\|^2_{\nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top}$$

$\Rightarrow$ We may still GD update, but the step size should be "***data-dependent***".

Intuitively, step size should be stretched hetrogeneously in different directions, being smaller when $\nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top$ is "larger".

ONS for Exp-concave Function

Online Newton Step

Input: parameters $\gamma, \varepsilon > 0$, matrix $A_0 = \varepsilon I_d$

At each round $t = 1, 2, \cdots$

(1) the player first picks a model $\mathbf{x}_t \in \mathcal{X} \subseteq \mathbb{R}^d$;

(2) and simultaneously environments pick an ***exp-concave loss function*** $f_t : \mathcal{X} \to \mathbb{R}$;

(3) the player suffers loss $f_t(\mathbf{x}_t)$, observes the information (loss) f_t and update:

$$\text{Update } A_t = A_{t-1} + \nabla f_t(\mathbf{x}_t) \nabla f_t(\mathbf{x}_t)^\top$$

$$\text{Update } \mathbf{x}_{t+1} = \arg\min_{\mathbf{x} \in \mathcal{X}} \left\| \mathbf{x} - \left(\mathbf{x}_t - \frac{1}{\gamma} A_t^{-1} \nabla f_t(\mathbf{x}_t) \right) \right\|_{A_t}^2$$

ONS: In a View of Proximal Gradient

Convex Problem

Property: $f_t(\mathbf{x}) \geq f_t(\mathbf{y}) + \nabla f_t(\mathbf{y})^\top (\mathbf{x} - \mathbf{y})$

OGD: $\mathbf{x}_{t+1} = \Pi_{\mathcal{X}} \left[\mathbf{x}_t - \frac{1}{\sqrt{t}} \nabla f_t(\mathbf{x}_t) \right]$

Proximal type update:

$$\mathbf{x}_{t+1} = \arg\min_{\mathbf{x} \in \mathcal{X}} \langle \mathbf{x}, \nabla f_t(\mathbf{x}_t) \rangle + \frac{1}{2\eta_t} \|\mathbf{x} - \mathbf{x}_t\|_2^2$$

Exp-concave Problem

Property: $f_t(\mathbf{x}) \geq f_t(\mathbf{y}) + \nabla f_t(\mathbf{y})^\top (\mathbf{x} - \mathbf{y}) + \frac{\gamma}{2} \|\mathbf{x} - \mathbf{y}\|^2_{\nabla f_t(\mathbf{y}) \nabla f_t(\mathbf{y})^\top}$

ONS: $A_t = A_{t-1} + \nabla f_t(\mathbf{x}_t) \nabla f_t(\mathbf{x}_t)^\top$

$$\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}^{A_t} \left[\mathbf{x}_t - \frac{1}{\gamma} A_t^{-1} \nabla f_t(\mathbf{x}_t) \right]$$

Proximal type update:

$$\mathbf{x}_{t+1} = \arg\min_{\mathbf{x} \in \mathcal{X}} \langle \mathbf{x}, \nabla f_t(\mathbf{x}_t) \rangle + \frac{\gamma}{2} \|\mathbf{x} - \mathbf{x}_t\|_{A_t}^2$$

ONS: In a View of Proximal Gradient

Proof.

$$\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}^{A_t}\left[\mathbf{x}_t - \frac{A_t^{-1}}{\gamma}\mathbf{g}_t\right] \qquad (\mathbf{g}_t \triangleq \nabla f_t(\mathbf{x}_t))$$

$$= \arg\min_{\mathbf{x}\in\mathcal{X}} \left(\mathbf{x} - \mathbf{x}_t + \frac{A_t^{-1}}{\gamma}\mathbf{g}_t\right)^\top A_t \left(\mathbf{x} - \mathbf{x}_t + \frac{A_t^{-1}}{\gamma}\mathbf{g}_t\right)$$

$$= \arg\min_{\mathbf{x}\in\mathcal{X}} \left(\mathbf{x} - \mathbf{x}_t + \frac{A_t^{-1}}{\gamma}\mathbf{g}_t\right)^\top \left(A_t\mathbf{x} - A_t\mathbf{x}_t + \frac{\mathbf{g}_t}{\gamma}\right)$$

$$= \arg\min_{\mathbf{x}\in\mathcal{X}} (\mathbf{x} - \mathbf{x}_t)^\top A_t (\mathbf{x} - \mathbf{x}_t) + \cancel{(A^{-1})^\top \mathbf{g}_t^\top \mathbf{g}_t} + 2\frac{\mathbf{g}_t^\top(\mathbf{x} - \cancel{\mathbf{x}_t})}{\gamma} \quad \text{(constant)}$$

$$= \arg\min_{\mathbf{x}\in\mathcal{X}} \langle \mathbf{x}, \mathbf{g}_t\rangle + \frac{\gamma}{2}\|\mathbf{x} - \mathbf{x}_t\|_{A_t}^2$$

Exp-concave Problem

Property: $f_t(\mathbf{x}) \geq f_t(\mathbf{y}) + \nabla f_t(\mathbf{y})^\top(\mathbf{x} - \mathbf{y}) + \frac{\gamma}{2}\|\mathbf{x} - \mathbf{y}\|^2_{\nabla f_t(\mathbf{y})\nabla f_t(\mathbf{y})^\top}$

ONS: $A_t = A_{t-1} + \nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top$

$$\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}^{A_t}\left[\mathbf{x}_t - \frac{1}{\gamma}A_t^{-1}\nabla f_t(\mathbf{x}_t)\right]$$

Proximal type update:

$$\mathbf{x}_{t+1} = \arg\min_{\mathbf{x}\in\mathcal{X}} \langle \mathbf{x}, \nabla f_t(\mathbf{x}_t)\rangle + \frac{\gamma}{2}\|\mathbf{x} - \mathbf{x}_t\|_{A_t}^2$$

ONS for Exp-concave Function

Theorem 5. *Under Assumptions 1, 2 and 3, for α-exp-concave online functions, the ONS algorithm with parameters $\gamma = \frac{1}{2}\min\left\{\frac{1}{GD}, \alpha\right\}$ and $\varepsilon = \frac{1}{\gamma^2 D^2}$ (recall that the initial matrix is $A_0 = \varepsilon I_d$) guarantees*

$$\text{Regret}_T \leq \mathcal{O}\left(\left(\frac{1}{\alpha} + GD\right) d \log T\right) = \mathcal{O}(d \log T),$$

where d is the dimension of the feasible domain $\mathcal{X} \subseteq \mathbb{R}^d$.

Proof

- $A_t = A_{t-1} + \nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top$
- $\mathbf{x}_{t+1} = \arg\min_{\mathbf{x}\in\mathcal{X}} \left\|\mathbf{x} - (\mathbf{x}_t - \frac{1}{\gamma}A_t^{-1}\mathbf{g}_t)\right\|_{A_t}^2$

Extending *the first GD lemma* to ***exp-concave case***:

Proof.

We use norm induced by A_t instead of 2-norm.

$$\|\mathbf{x}_{t+1} - \mathbf{u}\|_{A_t}^2 = \left\|\Pi_{\mathcal{X}}^{A_t}\left[\mathbf{x}_t - \frac{1}{\gamma}A_t^{-1}\nabla f_t(\mathbf{x}_t)\right] - \mathbf{u}\right\|_{A_t}^2 \qquad \left(\Pi_{\mathcal{X}}^{A}[\mathbf{y}] \triangleq \arg\min_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x}-\mathbf{y}\|_A^2\right)$$

$$\leq \left\|\mathbf{x}_t - \frac{1}{\gamma}A_t^{-1}\nabla f_t(\mathbf{x}_t) - \mathbf{u}\right\|_{A_t}^2$$

(A_t is semidefinite matrix)
(Pythagoras theorem)

$$= \left(\mathbf{x}_t - \frac{1}{\gamma}A_t^{-1}\nabla f_t(\mathbf{x}_t) - \mathbf{u}\right)^\top A_t \left(\mathbf{x}_t - \frac{1}{\gamma}A_t^{-1}\nabla f_t(\mathbf{x}_t) - \mathbf{u}\right)$$

(definition of $\|\cdot\|_{A_t}^2$)

$$= \left(\mathbf{x}_t - \mathbf{u} - \frac{1}{\gamma}A_t^{-1}\nabla f_t(\mathbf{x}_t)\right)^\top \left(A_t(\mathbf{x}_t - \mathbf{u}) - \frac{1}{\gamma}\nabla f_t(\mathbf{x}_t)\right)$$

Proof

- $A_t = A_{t-1} + \nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top$
- $\mathbf{x}_{t+1} = \arg\min_{\mathbf{x}\in\mathcal{X}} \left\| \mathbf{x} - (\mathbf{x}_t - \frac{1}{\gamma}A_t^{-1}\mathbf{g}_t) \right\|_{A_t}^2$

Extending *the first GD lemma* to ***exp-concave case***:

Proof.

$$\|\mathbf{x}_{t+1} - \mathbf{u}\|_{A_t}^2 = \left(\mathbf{x}_t - \mathbf{u} - \frac{1}{\gamma}A_t^{-1}\nabla f_t(\mathbf{x}_t)\right)^\top \left(A_t(\mathbf{x}_t - \mathbf{u}) - \frac{1}{\gamma}\nabla f_t(\mathbf{x}_t)\right)$$

$$= (\mathbf{x}_t - \mathbf{u})^\top A_t (\mathbf{x}_t - \mathbf{u}) - \frac{2}{\gamma}\nabla f_t(\mathbf{x}_t)^\top(\mathbf{x}_t - \mathbf{u}) + \frac{1}{\gamma^2}\nabla f_t(\mathbf{x}_t)^\top A_t^{-1}\nabla f_t(\mathbf{x}_t)$$

$$\leq \|\mathbf{x}_t - \mathbf{u}\|_{A_t}^2 - \frac{2}{\gamma}\left(f_t(\mathbf{x}_t) - f_t(\mathbf{u})\right) + \frac{1}{\gamma^2}\|\nabla f_t(\mathbf{x}_t)\|_{A_t^{-1}}^2$$

$$- (\mathbf{x}_t - \mathbf{u})^\top \nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top(\mathbf{x}_t - \mathbf{u})$$

(Exp-concave: $f_t(\mathbf{x}) \geq f_t(\mathbf{y}) + \nabla f_t(\mathbf{y})^\top(\mathbf{x}-\mathbf{y}) + \frac{\gamma}{2}(\mathbf{x}-\mathbf{y})^\top\nabla f_t(\mathbf{y})\nabla f_t(\mathbf{y})^\top(\mathbf{x}-\mathbf{y})$)

Proof

Proof. $\|\mathbf{x}_{t+1}-\mathbf{u}\|_{A_t}^2$

$$\leq \|\mathbf{x}_t-\mathbf{u}\|_{A_t}^2 - \frac{2}{\gamma}\left(f_t(\mathbf{x}_t)-f_t(\mathbf{u})\right) - \|\mathbf{x}_t-\mathbf{u}\|^2_{\nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top} + \frac{1}{\gamma^2}\|\nabla f_t(\mathbf{x}_t)\|^2_{A_t^{-1}}$$

$$\Longrightarrow f_t(\mathbf{x}_t)-f_t(\mathbf{u}) \leq \frac{\gamma}{2}\|\mathbf{x}_t-\mathbf{u}\|_{A_t}^2 - \frac{\gamma}{2}\|\mathbf{x}_{t+1}-\mathbf{u}\|_{A_t}^2 - \frac{\gamma}{2}\|\mathbf{x}_t-\mathbf{u}\|^2_{\nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top} + \frac{1}{2\gamma}\|\nabla f_t(\mathbf{x}_t)\|^2_{A_t^{-1}}$$

(rearranging)

Summing from $t=1$ to T, by telescoping:

$$\sum_{t=1}^T \left(f_t(\mathbf{x}_t)-f_t(\mathbf{u})\right) \leq \frac{\gamma}{2}\sum_{t=1}^T\left(\|\mathbf{x}_t-\mathbf{u}\|_{A_t}^2 - \|\mathbf{x}_t-\mathbf{u}\|_{A_{t-1}}^2\right) + \frac{1}{2\gamma}\sum_{t=1}^T\|\nabla f_t(\mathbf{x}_t)\|^2_{A_t^{-1}}$$

$$+\frac{\gamma}{2}\|\mathbf{x}_1-\mathbf{u}\|_{A_0}^2 - \frac{\gamma}{2}\sum_{t=1}^T\|\mathbf{x}_t-\mathbf{u}\|^2_{\nabla f(\mathbf{x}_t)\nabla f(\mathbf{x}_t)^\top}$$

cancellation

$$\leq \frac{\gamma}{2}\|\mathbf{x}_1-\mathbf{u}\|_{A_0}^2 + \frac{1}{2\gamma}\sum_{t=1}^T\|\nabla f_t(\mathbf{x}_t)\|^2_{A_t^{-1}}$$

$(A_t = A_{t-1} + \nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top)$

Proof

Proof.
$$\sum_{t=1}^{T}(f_t(\mathbf{x}_t) - f_t(\mathbf{u})) \leq \frac{\gamma}{2}\|\mathbf{x}_1 - \mathbf{u}\|_{A_0}^2 + \frac{1}{2\gamma}\sum_{t=1}^{T}\|\nabla f_t(\mathbf{x}_t)\|_{A_t^{-1}}^2$$

By the definition that $A_0 \triangleq \varepsilon I_d$, $\varepsilon = \frac{1}{\gamma^2 D^2}$ and the diameter $\|\mathbf{x}_1 - \mathbf{u}\|_2^2 \leq D^2$:

$$\sum_{t=1}^{T}(f_t(\mathbf{x}_t) - f_t(\mathbf{u})) \leq \frac{\gamma}{2}(\mathbf{x}_1 - \mathbf{u})^\top A_0 (\mathbf{x}_1 - \mathbf{u}) + \frac{1}{2\gamma}\sum_{t=1}^{T}\|\nabla f_t(\mathbf{x}_t)\|_{A_t^{-1}}^2$$

$$\leq \frac{1}{2\gamma} + \frac{1}{2\gamma}\sum_{t=1}^{T}\|\nabla f_t(\mathbf{x}_t)\|_{A_t^{-1}}^2 .$$

Next, we bound the term $\sum_{t=1}^{T}\|\nabla f_t(\mathbf{x}_t)\|_{A_t^{-1}}^2$.

Proof

Proof. Next, we bound the term $\sum_{t=1}^{T} \|\nabla f_t(\mathbf{x}_t)\|_{A_t^{-1}}^2$.

Lemma 4 **(Elliptical Potential Lemma).** *For any sequence* $\{X_1, \ldots, X_T\} \in \mathbb{R}^{d\times T}$*, suppose* $U_0 = \lambda I, U_t = U_{t-1} + X_t X_t^\top$*, and* $\|X_t\|_2 \leq L$*, then*

$$\sum_{t=1}^{T} \|X_t\|_{U_t^{-1}}^2 \leq d \log\left(1 + \frac{L^2 T}{\lambda d}\right)$$

Proof.

$$U_{t-1} = U_t - X_t X_t^\top = U_t^{\frac{1}{2}} \left(I - U_t^{-\frac{1}{2}} X_t X_t^\top U_t^{-\frac{1}{2}}\right) U_t^{\frac{1}{2}} \quad \text{(definition of } U_t\text{)}$$

$$\det(U_{t-1}) = \det(U_t) \det\left(I - U_t^{-\frac{1}{2}} X_t X_t^\top U_t^{-\frac{1}{2}}\right) \quad \text{(determinant on both side)}$$

Proof

Proof. Next, we bound the term $\sum_{t=1}^{T} \|\nabla f_t(\mathbf{x}_t)\|_{A_t^{-1}}^2$.

Lemma 5. *For any $\mathbf{v} \in \mathbb{R}^d$, we have*

$$\det\left(I - \mathbf{v}\mathbf{v}^\top\right) = 1 - \|\mathbf{v}\|_2^2$$

Proof.

(i) $\left(I - \mathbf{v}\mathbf{v}^\top\right)\mathbf{v} = \left(1 - \|\mathbf{v}\|_2^2\right)\mathbf{v}$, therefore, $\mathbf{v}$ is its eigenvector with $\left(1 - \|\mathbf{v}\|_2^2\right)$ as eigenvalue;

(ii) $\left(I - \mathbf{v}\mathbf{v}^\top\right)\mathbf{v}^\perp = \mathbf{v}^\perp$, therefore, $\mathbf{v}^\perp \perp \mathbf{v}$ is its eigenvector with 1 as the eigenvalue.

Proof

Proof. Next, we bound the term $\sum_{t=1}^{T} \|\nabla f_t(\mathbf{x}_t)\|_{A_t^{-1}}^2$.

Lemma 4 **(Elliptical Potential Lemma).** *For any sequence* $\{X_1, \ldots, X_T\} \in \mathbb{R}^{d\times T}$, *suppose* $U_0 = \lambda I, U_t = U_{t-1} + X_t X_t^\top$, *and* $\|X_t\|_2 \leq L$, *then*

$$\sum_{t=1}^{T} \|X_t\|_{U_t^{-1}}^2 \leq d \log\left(1 + \frac{L^2 T}{\lambda d}\right)$$

Proof. $\det(U_{t-1}) = \det(U_t)\det\left(I - U_t^{-\frac{1}{2}} X_t X_t^\top U_t^{-\frac{1}{2}}\right) = \det\left(U_t\right)\left(1 - \left\|U_t^{-\frac{1}{2}} X_t\right\|_2^2\right)$

(by Lemma 5)

$$\Longrightarrow \|X_t\|_{U_t^{-1}}^2 = \left\|U_t^{-\frac{1}{2}} X_t\right\|_2^2 = 1 - \frac{\det(U_{t-1})}{\det(U_t)}$$

(rearranging, U is a symmetric matrix)

Proof

Proof. Next, we bound the term $\sum_{t=1}^{T} \|\nabla f_t(\mathbf{x}_t)\|_{A_t^{-1}}^2$.

Lemma 4 (Elliptical Potential Lemma). *For any sequence $\{X_1, \ldots, X_T\} \in \mathbb{R}^{d\times T}$, suppose $U_0 = \lambda I, U_t = U_{t-1} + X_t X_t^\top$, and $\|X_t\|_2 \le L$, then*

$$\sum_{t=1}^{T} \|X_t\|_{U_t^{-1}}^2 \le d \log\left(1 + \frac{L^2 T}{\lambda d}\right)$$

Proof.

$$\Rightarrow \quad \sum_{t=1}^{T} X_t^\top U_t^{-1} X_t = \sum_{t=1}^{T} \left(1 - \frac{\det(U_{t-1})}{\det(U_t)}\right) \le \sum_{t=1}^{T} \log \frac{\det(U_t)}{\det(U_{t-1})} \quad (\forall x > 0, 1 - x \le -\log x)$$

$$= \log \frac{\det(U_T)}{\det(U_0)} = d \log\left(1 + \frac{L^2 T}{\lambda d}\right)$$

$\mathrm{Tr}(U_T) \le \mathrm{Tr}(U_0) + L^2 T = \lambda d + L^2 T$

$\Rightarrow \quad \det(U_T) \le (\lambda + L^2 T / d)^d$

Proof

Proof. Next, we bound the term $\sum_{t=1}^{T} \|\nabla f_t(\mathbf{x}_t)\|_{A_t^{-1}}^2$.

Lemma 4 (Elliptical Potential Lemma). *For any sequence* $\{X_1, \ldots, X_T\} \in \mathbb{R}^{d \times T}$*, suppose* $U_0 = \lambda I, U_t = U_{t-1} + X_t X_t^\top$*, and* $\|X_t\|_2 \leq L$*, then*

$$\sum_{t=1}^{T} \|X_t\|_{U_t^{-1}}^2 \leq d \log\left(1 + \frac{L^2 T}{\lambda d}\right)$$

Therefore, by Lemma 4, we have

$$\sum_{t=1}^{T} \|\nabla f_t(\mathbf{x}_t)\|_{A_t^{-1}}^2 \leq d \log\left(1 + \frac{D^2 T}{\varepsilon d}\right).$$

Proof

Proof. To conclude,

$$\sum_{t=1}^{T}(f_t(\mathbf{x}_t) - f_t(\mathbf{u})) \leq \underbrace{\frac{\gamma}{2}\|\mathbf{x}_1 - \mathbf{u}\|_{A_0}^2}_{\leq \frac{1}{2\gamma}} + \underbrace{\frac{1}{2\gamma}\sum_{t=1}^{T}\|\nabla f_t(\mathbf{x}_t)\|_{A_t^{-1}}^2}_{\leq \frac{d}{2\gamma}\log\left(1+\frac{D^2 T}{\varepsilon d}\right)}.$$

(bounded domain) (elliptical potential lemma)

Recall that $\gamma = \frac{1}{2}\min\left\{\frac{1}{GD}, \alpha\right\}$ and $\varepsilon = \frac{1}{\gamma^2 D^2}$,

$$\text{Regret}_T \leq \mathcal{O}\left((\frac{1}{\alpha} + GD)d\log T\right).$$

□

Lower Bounds

- A natural question: whether previous regret can be improved?

- Lower bound argument:

minimax bound: smallest possible worst-case regret of any algorithm:

$$\min_{\mathcal{A}} \max_{\ell_1,\ldots,\ell_T} \mathrm{Regret}_T$$

Theorem 7 (Lower Bound for OCO). *Any algorithm for online convex optimization incurs $\Omega(DG\sqrt{T})$ regret in the worst case. This is true even if the cost functions are generated from a fixed stationary distribution.*

Lower Bounds

Theorem 7 (Lower Bound for OCO). *Any algorithm for online convex optimization incurs $\Omega(DG\sqrt{T})$ regret in the worst case. This is true even if the cost functions are generated from a fixed stationary distribution.*

Proof Sketch.

Construct a "hard" environment:

- Binary classification, loss functions in each iteration are chosen at random
- Similar results can be obtained for strongly convex and exp-concave cases

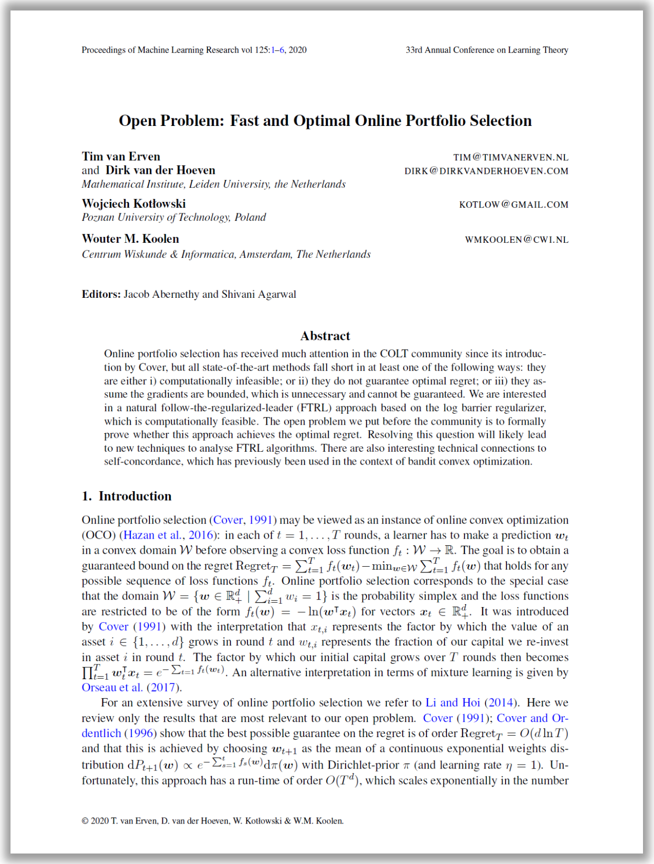
Comparison

	Algorithm	**Upper Bound**	**Lower Bound**
Convex	OGD	$\mathcal{O}(\sqrt{T})$	$\Omega(\sqrt{T})$
σ-Strongly Convex	OGD	$\mathcal{O}(\frac{\log T}{\sigma})$	$\Omega(\frac{\log T}{\sigma})$
α-Exp-concave	ONS	$\mathcal{O}(\frac{d \log T}{\alpha})$	$\Omega(\frac{d \log T}{\alpha})$

Back to Exp-concave Learning

- Universal Portfolio Selection

Algorithm	Regret	Runtime (per round)
Universal Portfolios	$d\log(T)$	d^4T^{14}
Online Gradient Descent	$G_2\sqrt{T}$	d
Exponentiated Gradient	$G_\infty\sqrt{T\log(d)}$	d
Online Newton Step (ONS)	$G_\infty d\log(T)$	d^2+generalized projection on Δ_d
Soft-Bayes	$\sqrt{dT\log(d)}$	d
Ada-BARRONS	$d^2\log^4(T)$	$d^{2.5}T$
BISONS	$d^2\log^2(T)$	$\mathrm{poly}(d)$
AdaMix+DONS	$d^2\log^5(T)$	d^3
VB-FTRL	$d\log(T)$	d^2T



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Open Problem: Fast and Optimal Online Portfolio Selection

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Editors: Jacob Abernethy and Shivani Agarwal

Abstract

Online portfolio selection has received much attention in the COLT community since its introduction by Cover, but all state-of-the-art methods fall short in at least one of the following ways: they are either i) computationally infeasible; or ii) they do not guarantee optimal regret; or iii) they assume the gradients are bounded, which is unnecessary and cannot be guaranteed. We are interested in a natural follow-the-regularized-leader (FTRL) approach based on the log barrier regularizer, which is computationally feasible. The open problem we put before the community is to formally prove whether this approach achieves the optimal regret. Resolving this question will likely lead to new techniques to analyse FTRL algorithms. There are also interesting technical connections to self-concordance, which has previously been used in the context of bandit convex optimization.

1. Introduction

Online portfolio selection (Cover, 1991) may be viewed as an instance of online convex optimization (OCO) (Hazan et al., 2016): in each of $t = 1, \dots, T$ rounds, a learner has to make a prediction $\boldsymbol{w}_t$ in a convex domain $\mathcal{W}$ before observing a convex loss function $f_t : \mathcal{W} \to \mathbb{R}$. The goal is to obtain a guaranteed bound on the regret $\mathrm{Regret}_T = \sum_{t=1}^T f_t(\boldsymbol{w}_t) - \min_{\boldsymbol{w}\in\mathcal{W}} \sum_{t=1}^T f_t(\boldsymbol{w})$ that holds for any possible sequence of loss functions f_t. Online portfolio selection corresponds to the special case that the domain $\mathcal{W} = \{\boldsymbol{w} \in \mathbb{R}_+^d \mid \sum_{i=1}^d w_i = 1\}$ is the probability simplex and the loss functions are restricted to be of the form $f_t(\boldsymbol{w}) = -\ln(\boldsymbol{w}^\intercal \boldsymbol{x}_t)$ for vectors $\boldsymbol{x}_t \in \mathbb{R}_+^d$. It was introduced by Cover (1991) with the interpretation that $x_{t,i}$ represents the factor by which the value of an asset $i \in \{1, \dots, d\}$ grows in round t and $w_{t,i}$ represents the fraction of our capital we re-invest in asset i in round t. The factor by which our initial capital grows over T rounds then becomes $\prod_{t=1}^T \boldsymbol{w}_t^\intercal \boldsymbol{x}_t = e^{-\sum_{t=1} f_t(\boldsymbol{w}_t)}$. An alternative interpretation in terms of mixture learning is given by Orseau et al. (2017).

For an extensive survey of online portfolio selection we refer to Li and Hoi (2014). Here we review only the results that are most relevant to our open problem. Cover (1991); Cover and Ordentlich (1996) show that the best possible guarantee on the regret is of order $\mathrm{Regret}_T = O(d\ln T)$ and that this is achieved by choosing $\boldsymbol{w}_{t+1}$ as the mean of a continuous exponential weights distribution $\mathrm{d}P_{t+1}(\boldsymbol{w}) \propto e^{-\sum_{s=1}^t f_s(\boldsymbol{w})}\mathrm{d}\pi(\boldsymbol{w})$ with Dirichlet-prior π (and learning rate $\eta = 1$). Unfortunately, this approach has a run-time of order $O(T^d)$, which scales exponentially in the number

[COLT 2020 Open Problem]

⇨ still an important open problem: efficiency and optimality

Part 3. Connection with Offline Learning

- Application to Stochastic Optimization

- Online-to-Batch Conversion

Application to Stochastic Optimization

- Consider the following ***convex optimization*** problem:

$$\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x})$$

- Stochastic optimization method

Computational oracle: only access ***noisy*** gradient oracle, namely, $\mathbf{g}(\mathbf{x})$, such that

$$\mathbb{E}[\mathbf{g}(\mathbf{x})] = \nabla f(\mathbf{x}), \text{ and } \mathbb{E}[\|\mathbf{g}(\mathbf{x})]\|^2] \leq G^2$$

for some $G > 0$.

Example (large-scale opt.). Given dataset $S = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)\}$, ERM optimizes

$$\min_{h\in\mathcal{H}} \sum_{i=1}^{m} \ell(h(\mathbf{x}_i), y_i) \Longrightarrow$$

full gradient computation requires a pass of ***all data***

stochastic method only uses a ***mini batch*** at each round

Stochastic Gradient Descent

- Consider the following ***convex optimization*** problem:

$$\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x})$$

Algorithm 2 Stochastic Gradient Descent

Input: noisy gradient oracle $\mathbf{g}(\cdot)$, step sizes $\{\eta_t\}$

1: **for** $t = 1, \ldots, T$ **do**

2: Obtain noisy gradient $\mathbf{g}(\mathbf{x}_t)$

3: Update the model $\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}\left[\mathbf{x}_t - \eta_t \mathbf{g}(\mathbf{x}_t)\right]$

4: **end for**

5: **return** $\bar{\mathbf{x}}_T = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t$

$$\mathbb{E}[\mathbf{g}(\mathbf{x})] = \nabla f(\mathbf{x})$$

$$\mathbb{E}[\|\mathbf{g}(\mathbf{x})]\|^2] \leq G^2$$

History: SGD

Robbins–Monro Method

A STOCHASTIC APPROXIMATION METHOD[1]

BY HERBERT ROBBINS AND SUTTON MONRO

University of North Carolina

1. Summary. Let $M(x)$ denote the expected value at level x of the response to a certain experiment. $M(x)$ is assumed to be a monotone function of x but is unknown to the experimenter, and it is desired to find the solution $x = \theta$ of the equation $M(x) = \alpha$, where α is a given constant. We give a method for making successive experiments at levels $x_1, x_2, \cdots$ in such a way that x_n will tend to θ in probability.

2. Introduction. Let $M(x)$ be a given function and α a given constant such that the equation

(1) $$M(x) = \alpha$$

has a unique root $x = \theta$. There are many methods for determining the value of θ by successive approximation. With any such method we begin by choosing one or more values $x_1, \cdots, x_r$ more or less arbitrarily, and then successively obtain new values x_n as certain functions of the previously obtained $x_1, \cdots, x_{n-1}$, the values $M(x_1), \cdots, M(x_{n-1})$, and possibly those of the derivatives $M'(x_1), \cdots, M'(x_{n-1})$, etc. If

(2) $$\lim_{n\to\infty} x_n = \theta,$$

irrespective of the arbitrary initial values $x_1, \cdots, x_r$, then the method is effective for the particular function $M(x)$ and value α. The speed of the convergence in (2) and the ease with which the x_n can be computed determine the practical utility of the method.

We consider a stochastic generalization of the above problem in which the nature of the function $M(x)$ is unknown to the experimenter. Instead, we suppose that to each value x corresponds a random variable $Y = Y(x)$ with distribution function $Pr[Y(x) \le y] = H(y \mid x)$, such that

(3) $$M(x) = \int_{-\infty}^{\infty} y \, dH(y \mid x)$$

is the expected value of Y for the given x. Neither the exact nature of $H(y \mid x)$ nor that of $M(x)$ is known to the experimenter, but it is assumed that equation (1) has a unique root θ, and it is desired to estimate θ by making successive observations on Y at levels $x_1, x_2, \cdots$ determined sequentially in accordance with some definite experimental procedure. If (2) holds *in probability* irrespective of any arbitrary initial values $x_1, \cdots, x_r$, we shall, in conformity with usual statistical terminology, call the procedure *consistent* for the given $H(y \mid x)$ and value α.

[1] This work was supported in part by the Office of Naval Research.

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Herbert Ellis Robbins
(1915 - 2001)

Kiefer–Wolfowitz Method

STOCHASTIC ESTIMATION OF THE MAXIMUM OF A REGRESSION FUNCTION[1]

BY J. KIEFER AND J. WOLFOWITZ

Cornell University

1. Summary. Let $M(x)$ be a regression function which has a maximum at the unknown point θ. $M(x)$ is itself unknown to the statistician who, however, can take observations at any level x. This paper gives a scheme whereby, starting from an arbitrary point x_1, one obtains successively $x_2, x_3, \cdots$ such that x_n converges to θ in probability as $n \to \infty$.

2. Introduction. Let $H(y \mid x)$ be a family of distribution functions which depend on a parameter x, and let

(2.1) $$M(x) = \int_{-\infty}^{\infty} y \, dH(y \mid x).$$

We suppose that

(2.2) $$\int_{-\infty}^{\infty} (y - M(x))^2 \, dH(y \mid x) \le S < \infty,$$

and that $M(x)$ is strictly increasing for $x < \theta$, and $M(x)$ is strictly decreasing for $x > \theta$. Let $\{a_n\}$ and $\{c_n\}$ be infinite sequences of positive numbers such that

(2.3) $$c_n \to 0,$$

(2.4) $$\sum a_n = \infty,$$

(2.5) $$\sum a_n c_n < \infty,$$

(2.6) $$\sum a_n^2 c_n^{-2} < \infty.$$

(For example, $a_n = n^{-1}$, $c_n = n^{-1/3}$.)

We can now describe a recursive scheme as follows. Let z_1 be an arbitrary number. For all positive integral n we have

(2.7) $$z_{n+1} = z_n + a_n \frac{(y_{2n} - y_{2n-1})}{c_n},$$

where y_{2n-1} and y_{2n} are independent chance variables with respective distributions $H(y \mid z_n - c_n)$ and $H(y \mid z_n + c_n)$. Under regularity conditions on $M(x)$ which we shall state below we will prove that z_n converges stochastically to θ (as $n \to \infty$).

The statistical importance of this problem is obvious and need not be discussed. The stimulus for this paper came from the interesting paper by Robbins and Monro [1] (see also Wolfowitz [2]).

[1] Research under contract with the Office of Naval Research. Presented to the American Mathematical Society at New York on April 25, 1952.

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Jack Kiefer
(1924 - 1981)

Jacob Wolfowitz
(1910 - 1981)

A Stochastic Approximation Method.
Herbert Robbins, Sutton Monro
Ann. Math. Statist. 22(3): 400-407 (September, 1951).

Stochastic Estimation of the Maximum of a Regression Function
Jack Kiefer, Jacob Wolfowitz
Ann. Math. Statist. 23(3): 462-466 (September, 1952)

History: SGD

Herbert Ellis Robbins
(1915 - 2001)

Statistical Science
1986, Vol. 1, No. 2, 276–284

The Contributions of Herbert Robbins to Mathematical Statistics

Tze Leung Lai and David Siegmund

Herbert Robbins was born on January 12, 1915, in New Castle, Pennsylvania. In 1931 he entered Harvard College at the age of 16. Although his interests until then had been predominantly literary, he found himself increasingly attracted to mathematics under the influence of Marston Morse, who during many long conversations conveyed a vivid sense of the intellectual challenge of creative work in that field (cf. Page, 1984, p. 7). He received the A.B. summa cum laude in 1935, and the Ph.D. in 1938, also from Harvard. His thesis, in the field of combinatorial topology and written under the supervision of Hassler Whitney, was published in 1941 [3]. (Numbers in brackets refer to Robbins' bibliography at the end of this article.)

After graduation, Robbins worked for a year at the Institute for Advanced Study at Princeton as Marston Morse's assistant. He then spent the next three years at New York University as instructor in mathematics. He became nationally known in 1941 as the coauthor, North Carolina at Chapel Hill. Having read [7] and [10], and greatly impressed by Robbins' mathematical skills, Hotelling offered him the position of associate professor to teach measure theory and probability to the graduate students in the new department. Robbins accepted the position and spent the next six years at Chapel Hill. During this relatively short period Robbins not only studied and developed an increasingly deep interest in statistics, but he also made a number of profound contributions to his new field: complete convergence [12], compound decision theory [25], stochastic approximation [26], and the sequential design of experiments [28], to name a few.

After a Guggenheim Fellowship at the Institute for Advanced Study during 1952–1953, Robbins moved from Chapel Hill to Columbia University as professor and chairman of the Department of Mathematical Statistics. Since 1953, with the exception of the three years 1965–1968 spent at Minnesota, Purdue, Berkeley, and Michigan, he has been at Columbia, where he

Stochastic Gradient Descent

Theorem 7 (Convergence of SGD). *Suppose the domain $\mathcal{X} \subseteq \mathbb{R}^d$ has a diameter $D > 0$, and the noisy gradient oracle is unbiased and variance bounded by G^2. SGD with step size $\eta_t = \frac{D}{G\sqrt{t}}$ guarantees*

$$\mathbb{E}[f(\bar{\mathbf{x}}_T)] - \min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x}) \leq \frac{3GD}{2\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right),$$

where $\bar{\mathbf{x}}_T \triangleq \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t$ is the output of the SGD algorithm.

Proof of SGD Convergence

Proof. First, we rephrase SGD from lens of ***online convex optimization***.

To see this, we define linear function $h_t(\mathbf{x}) \triangleq \mathbf{g}_t^\top \mathbf{x}$, where $\mathbf{g}_t = \mathbf{g}(\mathbf{x}_t)$.

Claim: deploying OGD over the online functions $\{h_t(\mathbf{x})\}$ is equivalent to SGD proposed in the earlier page.

OGD:
$$\begin{aligned}\mathbf{x}_{t+1} &= \Pi_{\mathcal{X}}[\mathbf{x}_t - \eta_t \nabla h_t(\mathbf{x}_t)] \\ &= \Pi_{\mathcal{X}}[\mathbf{x}_t - \eta_t \mathbf{g}(\mathbf{x}_t)]\end{aligned}$$

Algorithm 2 Stochastic Gradient Descent

Input: noisy gradient oracle $\mathbf{g}(\cdot)$, step sizes $\{\eta_t\}$

1: **for** $t = 1, \dots, T$ **do**
2: Obtain noisy gradient $\mathbf{g}(\mathbf{x}_t)$
3: Update the model $\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}[\mathbf{x}_t - \eta_t \mathbf{g}(\mathbf{x}_t)]$
4: **end for**
5: **return** $\bar{\mathbf{x}}_T = \frac{1}{T}\sum_{t=1}^T \mathbf{x}_t$

Proof of SGD Convergence

Proof.

$$\mathbb{E}\left[f(\overline{\mathbf{x}}_T)\right] - f(\mathbf{x}^\star) \leq \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} f(\mathbf{x}_t)\right] - f(\mathbf{x}^\star) \qquad (\mathbf{x}^\star = \arg\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x})) \quad \text{(Jensen's inequality)}$$

$$\leq \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} \nabla f(\mathbf{x}_t)^\top (\mathbf{x}_t - \mathbf{x}^\star)\right] \qquad \text{(convexity)}$$

$$= \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} \mathbf{g}_t^\top (\mathbf{x}_t - \mathbf{x}^\star)\right]$$

Theorem 3 (Regret bound for OGD). *Under Assumption 1, 2 and 3, online gradient descent (OGD) with step sizes $\eta_t = \frac{D}{G\sqrt{t}}$ for $t \in [T]$ guarantees:*

$$\text{Regret}_T = \sum_{t=1}^{T} f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^{T} f_t(\mathbf{x}) \leq \frac{3}{2}GD\sqrt{T}.$$

Proof:

Define $\boldsymbol{\xi}_t \triangleq \nabla f(\mathbf{x}_t) - \mathbf{g}_t$. We know $\mathbb{E}[\boldsymbol{\xi}_t \mid \mathbf{x}_t] = 0$.

We have $\mathbb{E}[\nabla f(\mathbf{x}_t)^\top \mathbf{x}_t] = \mathbb{E}[\boldsymbol{\xi}_t^\top \mathbf{x}_t] + \mathbb{E}[\mathbf{g}_t^\top \mathbf{x}_t]$

$$\mathbb{E}[\boldsymbol{\xi}_t^\top \mathbf{x}_t] = \mathbb{E}[\mathbb{E}[\boldsymbol{\xi}_t^\top \mathbf{x}_t \mid \mathbf{x}_t]] = \mathbb{E}[\mathbb{E}[\mathbb{E}[\boldsymbol{\xi}_t \mid \mathbf{x}_t]^\top \mathbf{x}_t \mid \mathbf{x}_t]] = 0.$$

Therefore, we have proved that $\mathbb{E}[\nabla f(\mathbf{x}_t)^\top \mathbf{x}_t] = \mathbb{E}[\mathbf{g}_t^\top \mathbf{x}_t]$.

Similar argument shows $\mathbb{E}[\nabla f(\mathbf{x}_t)^\top \mathbf{x}] = \mathbb{E}[\mathbf{g}_t^\top \mathbf{x}]$ for any *fixed* $\mathbf{x}$. □

Proof of SGD Convergence

Proof.

$$\mathbb{E}\left[f(\overline{\mathbf{x}}_T)\right] - f(\mathbf{x}^\star) \le \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} f(\mathbf{x}_t)\right] - f(\mathbf{x}^\star)$$

($\mathbf{x}^\star = \arg\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x})$)
(Jensen's inequality)

$$\le \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} \nabla f(\mathbf{x}_t)^\top (\mathbf{x}_t - \mathbf{x}^\star)\right]$$

(convexity)

$$= \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} \mathbf{g}_t^\top (\mathbf{x}_t - \mathbf{x}^\star)\right]$$

$$= \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} h_t(\mathbf{x}_t) - h_t(\mathbf{x}^\star)\right]$$

(definition of $f_t(\cdot)$)

$$\le \frac{\text{Regret}_T}{T}$$

(SGD = OGD over $\{f_t(\cdot)\}$)
(regret bound of OGD)

$$\le \frac{3GD}{2\sqrt{T}} \qquad \square$$

(regret of OGD algorithm)

Theorem 3 (Regret bound for OGD). *Under Assumption 1, 2 and 3, online gradient descent (OGD) with step sizes $\eta_t = \frac{D}{G\sqrt{t}}$ for $t \in [T]$ guarantees:*

$$\text{Regret}_T = \sum_{t=1}^{T} f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^{T} f_t(\mathbf{x}) \le \frac{3}{2}GD\sqrt{T}.$$

Stochastic Gradient Descent

Theorem 7 (Convergence of SGD). *Suppose the domain $\mathcal{X} \subseteq \mathbb{R}^d$ has a diameter $D > 0$, and the noisy gradient oracle is unbiased and variance bounded by G^2. SGD with step size $\eta_t = \frac{D}{G\sqrt{t}}$ guarantees*

$$\mathbb{E}[f(\bar{\mathbf{x}}_T)] \le \min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}) + \frac{3GD}{2\sqrt{T}},$$

where $\bar{\mathbf{x}}_T \triangleq \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t$ is the output of the SGD algorithm.

- We define the linear function $h_t(\mathbf{x}) \triangleq \mathbf{g}_t^\top \mathbf{x} = \mathbf{g}(\mathbf{x}_t)^\top \mathbf{x}$ and run OGD on $\{h_t\}_{t=1}^{T}$.
- Note that function h_t ***depends*** on the decision $\mathbf{x}_t$, which actually reveals that OGD regret can hold even against ***adaptive adversary***.

More bits of OGD

- We define the linear function $h_t(\mathbf{x}) \triangleq \mathbf{g}_t^\top \mathbf{x} = \mathbf{g}(\mathbf{x}_t)^\top \mathbf{x}$ and run OGD on $\{h_t\}_{t=1}^T$.
- Note that function h_t ***depends*** on the decision $\mathbf{x}_t$, which actually reveals that OGD regret can hold even against ***adaptive adversary***.

At each round $t = 1, 2, \cdots$

(1) the player first picks a model $\mathbf{x}_t \in \mathcal{X}$;

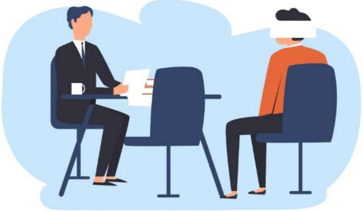
(2) and simultaneously environments pick an online function $f_t : \mathcal{X} \to \mathbb{R}$;

(3) the player suffers loss $f_t(\mathbf{x}_t)$, observes some information about f_t and updates the model.

oblivious adversary

examination

adaptive adversary



interview

The "simultaneously" requirement can be sometimes not necessary!

OGD for full-info OCO can handle the case when online functions depend on $\mathbf{x}_t$!

Online-to-Batch Conversion

- An alternative way to solve statistic learning:
  - use the data in a sequential way
  - run any online algorithm minimizing the regret
  - return the final model as the average

Algorithm 1 Online-to-Batch Conversion

Input: Data $\{(\mathbf{x}_1, y_1), \cdots, (\mathbf{x}_T, y_T)\}$ i.i.d. sampled from the distribution $\mathcal{D}$, a bounded loss function $\ell : \mathcal{Y} \times \mathcal{Y} \to [0, 1]$, an online learning algorithm $\mathcal{A}$

1: **for** $t = 1, \cdots, T$ **do**
2: let $\mathbf{w}_t$ be the output of algorithm $\mathcal{A}$ for this round
3: Feed algorithm $\mathcal{A}$ with loss function $f_t(\mathbf{w}) = \ell(\mathbf{w}^\top \mathbf{x}_t, y_t)$
4: **end for**
5: **return** $\bar{\mathbf{w}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{w}_t$

Online-to-Batch Conversion

Theorem 2 (Online-to-Batch Conversion). *If the risk $R(\mathbf{w})$ is convex w.r.t. $\mathbf{w}$ with a bounded loss function $\ell : \mathcal{Y} \times \mathcal{Y} \to [0,1]$, and the data $\{(\mathbf{x}_1, y_1), \cdots, (\mathbf{x}_T, y_T)\}$ are i.i.d. sampled from the distribution $\mathcal{D}$, then with probability at least $1-\delta$, the excess risk of the output of Algorithm 1 satisfies*

$$R(\bar{\mathbf{w}}) - \min_{\mathbf{w}\in\mathcal{W}} R(\mathbf{w}) \leq \frac{\text{Regret}_T}{T} + 2\sqrt{\frac{2\log(2/\delta)}{T}}$$

where $R(\mathbf{w}) \triangleq \mathbb{E}_{(\mathbf{x},y)\sim\mathcal{D}}[\ell(h(\mathbf{w};\mathbf{x}), y)]$ is the expected risk, and $\text{Regret}_T \triangleq \sum_{t=1}^T f_t(\mathbf{w}_t) - \min_{\mathbf{w}\in\mathcal{W}} \sum_{t=1}^T f_t(\mathbf{w})$ is the regret of the online algorithm $\mathcal{A}$ after T rounds.

Concentration Inequalities

Lemma 1 (Hoeffding's inequality). *Let* $X_1, \ldots, X_T \in [-B, B]$ *for some* $B > 0$ *be independent random variables such that* $\mathbb{E}[X_t] = 0$ *for all* $t \in [T]$, *then for all* $\delta \in (0,1)$,

$$\Pr\left[\sum_{t=1}^{T} X_t \geq B\sqrt{2T \ln \frac{1}{\delta}}\right] \leq \delta$$

Lemma 2 (Azuma's inequality). *Let* $X_1, \ldots, X_T \in [-B, B]$ *for some* $B > 0$ *be a martingale difference sequence (i.e.,* $\forall t \in [T]$, $\mathbb{E}[X_t \mid X_{t-1}, \ldots, X_1] = 0$ *), then* $\forall \delta > 0$,

$$\Pr\left[\sum_{t=1}^{T} X_t \geq B\sqrt{2T \ln \frac{1}{\delta}}\right] \leq \delta$$

Online-to-Batch Conversion

Theorem 2 (Online-to-Batch Conversion). *If the risk $R(\mathbf{w})$ is convex w.r.t. $\mathbf{w}$ with a bounded loss function $\ell : \mathcal{Y} \times \mathcal{Y} \to [0,1]$, and the data $\{(\mathbf{x}_1, y_1), \cdots, (\mathbf{x}_T, y_T)\}$ are i.i.d. sampled from the distribution $\mathcal{D}$, then with probability at least $1-\delta$, the excess risk of the output of Algorithm 1 satisfies*

$$R(\bar{\mathbf{w}}) - \min_{\mathbf{w}\in\mathcal{W}} R(\mathbf{w}) \leq \frac{\mathrm{Regret}_T}{T} + 2\sqrt{\frac{2\ln(2/\delta)}{T}}$$

where $R(\mathbf{w}) \triangleq \mathbb{E}_{(\mathbf{x},y)\sim\mathcal{D}}[\ell(h(\mathbf{w};\mathbf{x}), y)]$ is the expected risk, and $\mathrm{Regret}_T \triangleq \sum_{t=1}^T f_t(\mathbf{w}_t) - \min_{\mathbf{w}\in\mathcal{W}} \sum_{t=1}^T f_t(\mathbf{w})$ is the regret of the online algorithm $\mathcal{A}$ after T rounds.

Proof Sketch.

$$R(\hat{\mathbf{w}}) \overset{\text{Jensen's inequality}}{\leq} \frac{1}{T}\sum_{t=1}^T R(\mathbf{w}_t) \overset{\text{Azuma's inequality}}{\leq} \frac{1}{T}\sum_{t=1}^T f_t(\mathbf{w}_t) + \sqrt{\frac{2\ln(2/\delta)}{T}}$$

$$\frac{1}{T}\sum_{t=1}^T f_t(\mathbf{w}_t) \overset{\frac{\mathrm{Regret}_T}{T}}{\Longleftrightarrow} \frac{1}{T}\min_{\mathbf{w}\in\mathcal{W}} \sum_{t=1}^T f_t(\mathbf{w})$$

$$R(\mathbf{w}^*) + \sqrt{\frac{2\ln(2/\delta)}{T}} \overset{\text{Hoeffding's inequality}}{\geq} \frac{1}{T}\sum_{t=1}^T f_t(\mathbf{w}^\star) \geq \frac{1}{T}\min_{\mathbf{w}\in\mathcal{W}} \sum_{t=1}^T f_t(\mathbf{w})$$

Online-to-Batch Conversion

Proof.

$$
\begin{aligned}
R(\hat{\mathbf{w}}) &= \mathbb{E}_{(\mathbf{x},y)\sim\mathcal{D}}\left[\ell(h(\hat{\mathbf{w}};\mathbf{x}),y)\right] \\
&= \mathbb{E}_{(\mathbf{x},y)\sim\mathcal{D}}\left[\ell\Big(h(\frac{1}{T}\sum_{t=1}^{T}\mathbf{w}_t;\mathbf{x}),y\Big)\right] \\
&\le \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}_{(\mathbf{x},y)\sim\mathcal{D}}\left[\ell(h(\mathbf{w}_t;\mathbf{x}),y)\right] && \text{(Jensen's inequality)} \\
&= \frac{1}{T}\sum_{t=1}^{T} f_t(\mathbf{w}_t) + \sqrt{\frac{2\ln(2/\delta)}{T}} && \text{(Azuma's inequality with } X_t = \mathbb{E}_{(\mathbf{x},y)\sim\mathcal{D}}\left[\ell(h(\mathbf{w}_t;\mathbf{x}),y)\right] - f_t(\mathbf{w}_t)\text{)}
\end{aligned}
$$

Online-to-Batch Conversion

Proof.

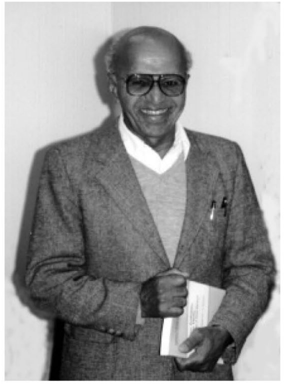
$$
\begin{aligned}
R(\hat{\mathbf{w}}) &\le \frac{1}{T}\sum_{t=1}^{T} f_t(\mathbf{w}_t) + \sqrt{\frac{2\ln(2/\delta)}{T}} \\
&= \min_{\mathbf{w}\in\mathcal{W}} \frac{1}{T}\sum_{t=1}^{T} f_t(\mathbf{w}) + \frac{\text{Regret}_T}{T} + \sqrt{\frac{2\ln(2/\delta)}{T}} && \text{(definition of regret)} \\
&\le \frac{1}{T}\sum_{t=1}^{T} f_t(\mathbf{w}^\star) + \frac{\text{Regret}_T}{T} + \sqrt{\frac{2\ln(2/\delta)}{T}} \\
&\le R(\mathbf{w}^\star) + \frac{\text{Regret}_T}{T} + 2\sqrt{\frac{2\ln(2/\delta)}{T}} && \text{(Hoeffding's inequality with } X_t = f_t(\mathbf{w}^\star) - R(\mathbf{w}^\star)\text{)}
\end{aligned}
$$

□

History: Two-Player Zero-Sum Games

Theory of repeated games

James Hannan
(1922–2010)



David Blackwell
(1919–2010)

Learning to play a game (1956)

Play a game repeatedly against a possibly suboptimal opponent

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Zero-sum 2-person games played more than once

	1	2	$\dots$	M
1	$\ell(1,1)$	$\ell(1,2)$	$\dots$	
2	$\ell(2,1)$	$\ell(2,2)$	$\dots$	
$\vdots$	$\vdots$	$\vdots$	$\ddots$	
N				

$N \times M$ known loss matrix

- Row player (player) has N actions
- Column player (opponent) has M actions

For each game round $t = 1, 2, \dots$

- Player chooses action i_t and opponent chooses action y_t
- The player suffers loss $\ell(i_t, y_t)$ (= gain of opponent)

Player can learn from opponent's history of past choices $y_1, \dots, y_{t-1}$

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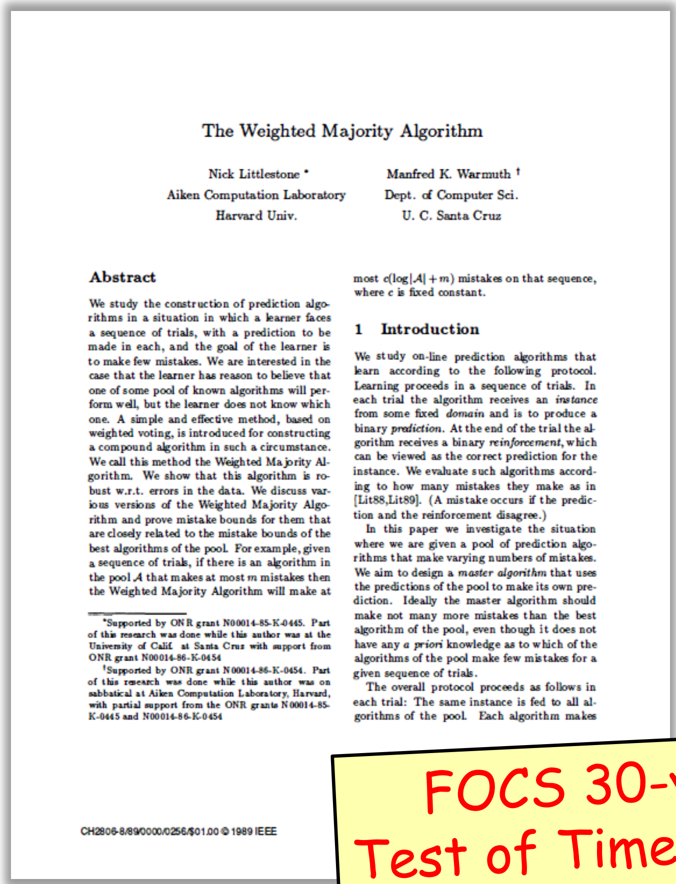
Nicolo Cesa-Bianchi, Online Learning and Online Convex Optimization. Tutorial at the Simons Institute. 2017.

History: Prediction with Expert Advice



The Weighted Majority Algorithm

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Abstract

We study the construction of prediction algorithms in a situation in which a learner faces a sequence of trials, with a prediction to be made in each, and the goal of the learner is to make few mistakes. We are interested in the case that the learner has reason to believe that one of some pool of known algorithms will perform well, but the learner does not know which one. A simple and effective method, based on weighted voting, is introduced for constructing a compound algorithm in such a circumstance. We call this method the Weighted Majority Algorithm. We show that this algorithm is robust w.r.t. errors in the data. We discuss various versions of the Weighted Majority Algorithm and prove mistake bounds for them that are closely related to the mistake bounds of the best algorithms of the pool. For example, given a sequence of trials, if there is an algorithm in the pool $\mathcal{A}$ that makes at most m mistakes then the Weighted Majority Algorithm will make at most $c(\log|\mathcal{A}| + m)$ mistakes on that sequence, where c is fixed constant.

1 Introduction

We study on-line prediction algorithms that learn according to the following protocol. Learning proceeds in a sequence of trials. In each trial the algorithm receives an *instance* from some fixed *domain* and is to produce a binary *prediction*. At the end of the trial the algorithm receives a binary *reinforcement*, which can be viewed as the correct prediction for the instance. We evaluate such algorithms according to how many mistakes they make as in [Lit88,Lit89]. (A mistake occurs if the prediction and the reinforcement disagree.)

In this paper we investigate the situation where we are given a pool of prediction algorithms that make varying numbers of mistakes. We aim to design a *master algorithm* that uses the predictions of the pool to make its own prediction. Ideally the master algorithm should make not many more mistakes than the best algorithm of the pool, even though it does not have any *a priori* knowledge as to which of the algorithms of the pool make few mistakes for a given sequence of trials.

The overall protocol proceeds as follows in each trial: The same instance is fed to all algorithms of the pool. Each algorithm makes

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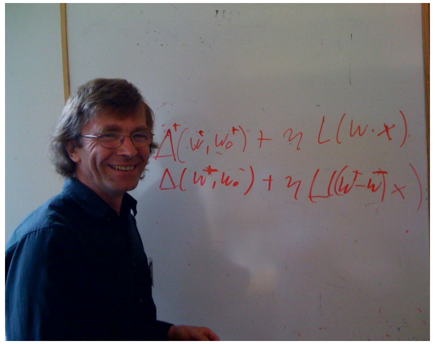
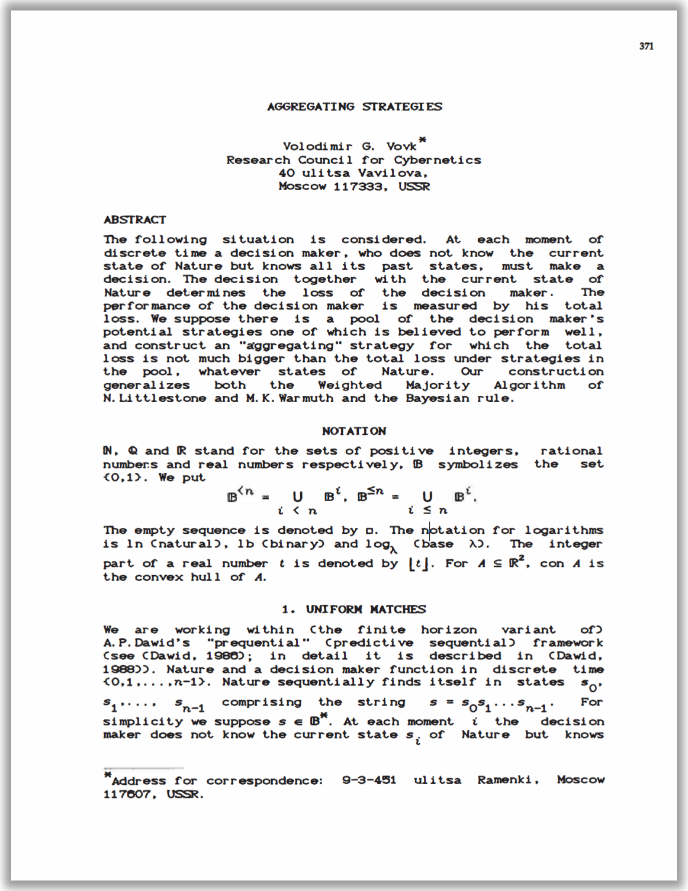
FOCS 30-year
Test of Time Award!



Manfred Warmuth

UC Santa Cruz

Nick Littlestone and Manfred K. Warmuth. "The Weighted Majority Algorithm." FOCS 1989: 256-261.



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AGGREGATING STRATEGIES

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ABSTRACT

The following situation is considered. At each moment of discrete time a decision maker, who does not know the current state of Nature but knows all its past states, must make a decision. The decision together with the current state of Nature determines the loss of the decision maker. The performance of the decision maker is measured by his total loss. We suppose there is a pool of the decision maker's potential strategies one of which is believed to perform well, and construct an "aggregating" strategy for which the total loss is not much bigger than the total loss under strategies in the pool, whatever states of Nature. Our construction generalizes both the Weighted Majority Algorithm of N. Littlestone and M. K. Warmuth and the Bayesian rule.

NOTATION

$\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ stand for the sets of positive integers, rational numbers and real numbers respectively, $\mathbb{B}$ symbolizes the set {0,1}. We put

$$\mathbb{B}^{<n} = \bigcup_{i<n} \mathbb{B}^i, \quad \mathbb{B}^{\le n} = \bigcup_{i \le n} \mathbb{B}^i.$$

The empty sequence is denoted by $\square$. The notation for logarithms is ln (natural), lb (binary) and $\log_\lambda$ (base λ). The integer part of a real number t is denoted by $\lfloor t \rfloor$. For $A \subseteq \mathbb{R}^2$, con A is the convex hull of A.

1. UNIFORM MATCHES

We are working within (the finite horizon variant of) A. P. Dawid's "prequential" (predictive sequential) framework (see (Dawid, 1986); in detail it is described in (Dawid, 1988)). Nature and a decision maker function in discrete time {0,1,...,n-1}. Nature sequentially finds itself in states s_0, s_1,..., s_{n-1} comprising the string $s = s_0 s_1 \ldots s_{n-1}$. For simplicity we suppose $s \in \mathbb{B}^*$. At each moment i the decision maker does not know the current state s_i of Nature but knows

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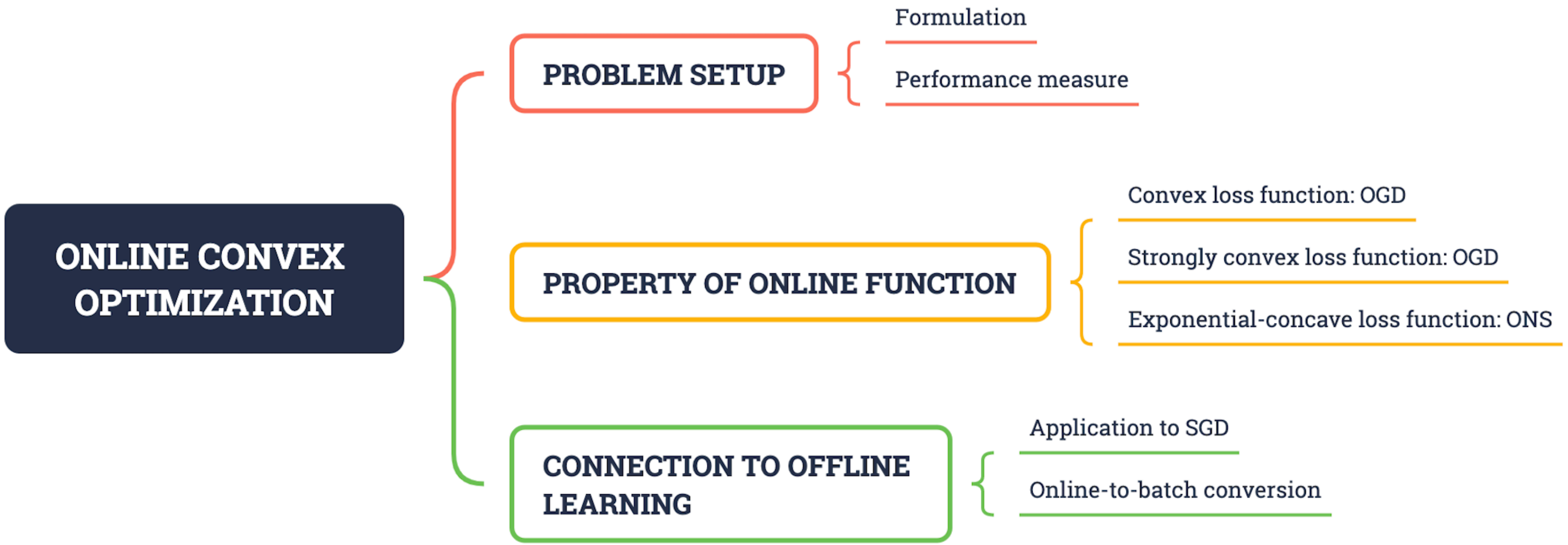
Volodimir G. Vovk. "Aggregating Strategies." COLT 1990: 371-383.

Summary



Q & A

Thanks!