Lecture 3. Gradient Descent Method

Advanced Optimization (Fall 2025)

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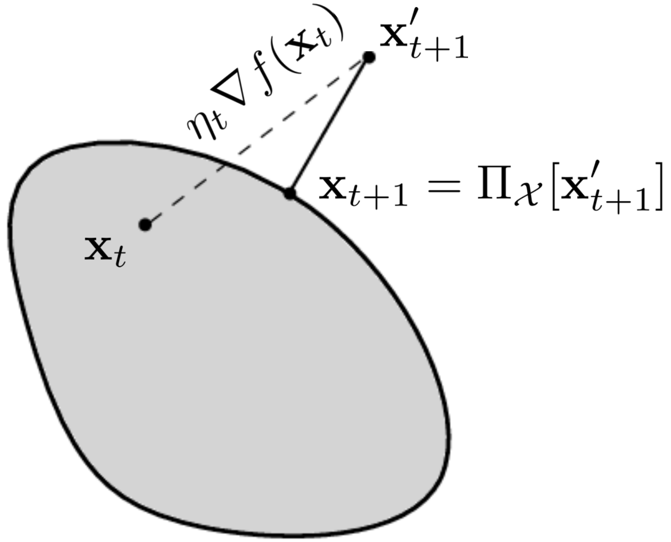
Outline

- Gradient Descent
- Convex and Lipschitz
  - Polyak Step Size
  - Convergence without Optimal Value
  - Optimal Time-Varying Step Sizes
- Strongly Convex and Lipschitz

Gradient Descent

- GD Template:

$$\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}\left[\mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t)\right]$$



  - $\mathbf{x}_1$ can be an arbitrary point inside the domain.
  - $\eta_t > 0$ is the potentially time-varying ***step size*** (or called ***learning rate***).
  - Projection $\Pi_{\mathcal{X}}[\mathbf{y}] = \arg\min_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x} - \mathbf{y}\|$ ensures the feasibility.

GD Convergence

- Lipschitz (non-smooth) optimization
  - Convex
  - Strongly convex
- Smooth optimization
  - Convex
  - Strongly convex

This lecture will focus on GD analysis for ***Lipschitz*** functions, and next lecture will discuss ***smooth*** functions.

The First Gradient Descent Lemma

Lemma 1. *Suppose that f is proper, closed and convex; the feasible domain $\mathcal{X}$ is nonempty, closed and convex. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by the gradient descent method, $\mathcal{X}^\star$ be the optimal set of the optimization problem and $f^\star$ be the optimal value. Then for any $\mathbf{x}^\star \in \mathcal{X}^\star$ and $t \geq 0$,*

$$\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - 2\eta_t(f(\mathbf{x}_t) - f^\star) + \eta_t^2\|\nabla f(\mathbf{x}_t)\|^2.$$

Proof:

$$\begin{aligned}
\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 &= \|\Pi_{\mathcal{X}}[\mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t)] - \mathbf{x}^\star\|^2 \quad \text{(GD)} \\
&\leq \|\mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t) - \mathbf{x}^\star\|^2 \quad \text{(Pythagoras Theorem)} \\
&= \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - 2\eta_t\langle \nabla f(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}^\star\rangle + \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2 \\
&\leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - 2\eta_t(f(\mathbf{x}_t) - f^\star) + \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2
\end{aligned}$$

(convexity: $f(\mathbf{x}_t) - f^\star = f(\mathbf{x}_t) - f(\mathbf{x}^\star) \leq \langle \nabla f(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}^\star\rangle$) $\square$

Part 1. Polyak Step Size

- Polyak Step Size

- Convergence

- Convergence Rate

Polyak Step Size

- GD method satisfies the following inequality:

$$\boxed{\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \le \|\mathbf{x}_t - \mathbf{x}^\star\|^2 \underbrace{- 2\eta_t(f(\mathbf{x}_t) - f^\star) + \eta_t^2\|\nabla f(\mathbf{x}_t)\|^2}}$$

$$h(\eta) \triangleq -2\eta(f(\mathbf{x}_t) - f^\star) + \eta^2\|\nabla f(\mathbf{x}_t)\|^2$$

A natural idea:

minimizing the right-hand side of the inequality

$$\Longrightarrow \quad \eta_t = \frac{f(\mathbf{x}_t) - f^\star}{\|\nabla f(\mathbf{x}_t)\|^2}$$

assume known $f^\star$ for a moment

Polyak Step Size

$$\boxed{\color{red}{\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \le \|\mathbf{x}_t - \mathbf{x}^\star\|^2} \underbrace{- 2\eta_t(f(\mathbf{x}_t) - f^\star) + \eta_t^2\|\nabla f(\mathbf{x}_t)\|^2}}$$

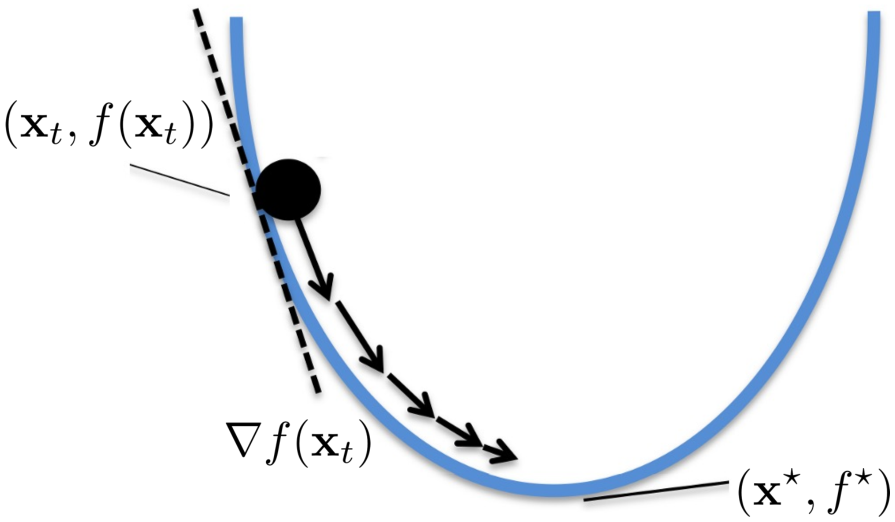
$$\color{blue}{h(\eta) \triangleq -2\eta(f(\mathbf{x}_t) - f^\star) + \eta^2\|\nabla f(\mathbf{x}_t)\|^2}$$

Cornercase: when $\nabla f(\mathbf{x}_t) = \mathbf{0}$

$\Longrightarrow$ actually a good news owing to convexity, $\nabla f(\mathbf{x}_t) = \mathbf{0}$ implies *optimality*

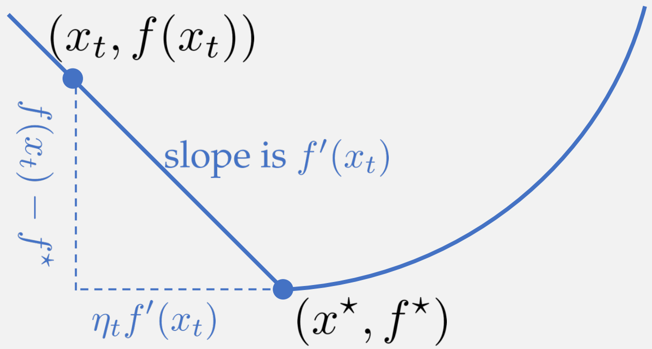
Polyak step size: $\eta_t = \begin{cases} \frac{f(\mathbf{x}_t) - f^\star}{\|\nabla f(\mathbf{x}_t)\|^2}, & \nabla f(\mathbf{x}_t) \neq \mathbf{0} \\ 1, & \nabla f(\mathbf{x}_t) = \mathbf{0} \end{cases}$

Polyak Step Size: A Geometric View



Q: if we have known $f^\star$ already, how would we set $\mathbf{x}_{t+1}$?

Geometric way to "optimize" (consider the 1-dim function)



Geometrically, the best way of iterates

$$x_{t+1} = x_t - \eta_t f'(x_t)$$

would satisfy that (given known $f^\star$)

$$\eta_t f'(x_t) \cdot f'(x_t) = f(x_t) - f^\star$$

$$\Rightarrow \quad \eta_t = \frac{f(x_t) - f^\star}{f'(x_t)^2}$$

(Unconstrained) GD with Polyak Step Size

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t), \quad \eta_t = \frac{f(\mathbf{x}_t) - f^\star}{\|\nabla f(\mathbf{x}_t)\|^2}$$

Convergence

- With Polyak step size, we obtain the convergence results.

Theorem 1. *Under the same assumptions with Lemma 1, assume the gradient of f is bounded by G, i.e., $\|\nabla f(\cdot)\| \leq G$. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by the gradient descent method with Polyak step size and $f^\star$ be the optimal value. Then,*

(i) $\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2$.

(ii) $f(\mathbf{x}_t) \to f^\star$ *as* $t \to \infty$.

Note: recall that ***bounded gradients*** condition implies ***Lipschitz continuity***.

Convergence

Proof: $\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - 2\eta_t(f(\mathbf{x}_t) - f^\star) + \eta_t^2\|\nabla f(\mathbf{x}_t)\|^2$

(the first GD lemma)

- **Case 1:** $\nabla f(\mathbf{x}_t) = \mathbf{0}$. By convexity, $f(\mathbf{x}_t) = f^\star \Rightarrow \|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 = \|\mathbf{x}_t - \mathbf{x}^\star\|^2$.
- **Case 2:** $\nabla f(\mathbf{x}_t) \neq \mathbf{0}$. Polyak's step size $\eta_t = \frac{f(\mathbf{x}_t) - f^\star}{\|\nabla f(\mathbf{x}_t)\|^2}$

$$\Longrightarrow \|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \frac{(f(\mathbf{x}_t) - f^\star)^2}{\|\nabla f(\mathbf{x}_t)\|^2} \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2$$

(*i*) is proved.

Convergence

Proof: we can simply focus on the case of $\nabla f(\mathbf{x}_t) \neq \mathbf{0}$

$$\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \frac{(f(\mathbf{x}_t) - f^\star)^2}{\|\nabla f(\mathbf{x}_t)\|^2} \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \frac{(f(\mathbf{x}_t) - f^\star)^2}{G^2}$$

$(\|\nabla f(\cdot)\| \leq G)$

$$\Longrightarrow \quad \frac{1}{G^2}\sum_{t=1}^{T}(f(\mathbf{x}_t) - f^\star)^2 \leq \|\mathbf{x}_1 - \mathbf{x}^\star\|^2 - \|\mathbf{x}_{T+1} - \mathbf{x}^\star\|^2$$

$$\Longrightarrow \quad \sum_{t=1}^{T}(f(\mathbf{x}_t) - f^\star)^2 \leq G^2\|\mathbf{x}_1 - \mathbf{x}^\star\|^2$$

Infinite summation is bounded by constants → **convergent** series.

***(ii)* is proved.** $\square$

Convergence Rate

Polyak step size: $\eta_t = \begin{cases} \frac{f(\mathbf{x}_t)-f^\star}{\|\nabla f(\mathbf{x}_t)\|^2}, & \nabla f(\mathbf{x}_t) \neq \mathbf{0} \\ 1, & \nabla f(\mathbf{x}_t) = \mathbf{0} \end{cases}$

- Theorem 1 proves the convergence, and now we give the ***rate***.

Theorem 2. *Assume the gradient of f is bounded by G, i.e., $\|\nabla f(\cdot)\| \leq G$. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by the GD method with Polyak step size and $f^\star$ be the optimal value. Define $\bar{\mathbf{x}}_T = \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$, we have*

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{G\|\mathbf{x}_1 - \mathbf{x}^\star\|}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right).$$

best-iterated guarantee

Proof:

$$\left.\begin{aligned} & f(\bar{\mathbf{x}}_T) = \min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t) \leq f(\mathbf{x}_t) \\ & \sum_{t=1}^T (f(\mathbf{x}_t) - f^\star)^2 \leq G^2\|\mathbf{x}_1 - \mathbf{x}^\star\|^2 \end{aligned}\right\} T(f(\bar{\mathbf{x}}_T) - f^\star)^2 \leq G^2\|\mathbf{x}_1 - \mathbf{x}^\star\|^2$$

$\square$

Optimality

- It can be proved that $O(1/\sqrt{T})$ is minimax optimal for first-order methods when optimizing **convex and Lipschitz functions**.

Theorem 3 (Bubeck, 2015, Theorem 3.13). *Let $t \le n$ and $L, R > 0$. We consider the class of first-order (black-box) procedures satisfying $\mathbf{x}_1 = \mathbf{0}$, and for any $t \ge 1$, $\mathbf{x}_{t+1} \in \mathrm{Span}(\mathbf{g}_1, \ldots, \mathbf{g}_t)$, where $\mathbf{g}_s = \nabla f(\mathbf{x}_s)$ is the gradient (or subgradient) queried from the oracle at $\mathbf{x}_s$.*

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the canonical basis of $\mathbb{R}^n$, and denote by $B_2(R) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_2 \le R\}$. Then there exists a convex and L-Lipschitz function $f : B_2(R) \to \mathbb{R}$ such that for any black-box procedure satisfying the above,

$$\min_{1 \le s \le t} f(\mathbf{x}_s) - \min_{\mathbf{x} \in B_2(R)} f(\mathbf{x}) \ge \frac{RL}{2(1+\sqrt{t})}.$$

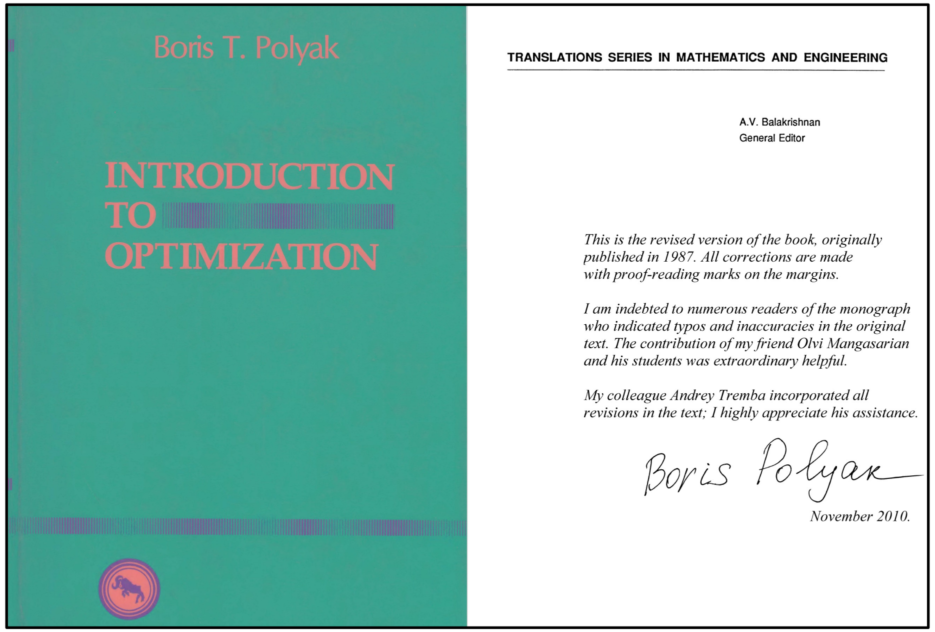
Why Polyak Step Size Matters

- Although Polyak step size [Polyak, 1969] looks simple and even requires knowing $f^\star$, it has a profound ***enlightening*** value.
- It established the foundation for the study of step-size tuning and convergence-rate analysis in modern optimization.

- Technically,
  - Optimal theoretical convergence with minimal assumptions
  - Connect the step size tuning with the gradient norms
  - Lays the conceptual ground for future developments: adaptive tuning, universal optimization, parameter-free optimization, etc.

Polyak Step Size

$$\textbf{\textit{Polyak step size:}} \quad \eta_t = \begin{cases} \frac{f(\mathbf{x}_t) - f^\star}{\|\nabla f(\mathbf{x}_t)\|^2}, & \nabla f(\mathbf{x}_t) \neq \mathbf{0} \\ 1, & \nabla f(\mathbf{x}_t) = \mathbf{0} \end{cases}$$


Boris T. Polyak

INTRODUCTION TO OPTIMIZATION

TRANSLATIONS SERIES IN MATHEMATICS AND ENGINEERING

A.V. Balakrishnan
General Editor

This is the revised version of the book, originally published in 1987. All corrections are made with proof-reading marks on the margins.

I am indebted to numerous readers of the monograph who indicated typos and inaccuracies in the original text. The contribution of my friend Olvi Mangasarian and his students was extraordinary helpful.

My colleague Andrey Tremba incorporated all revisions in the text; I highly appreciate his assistance.

Boris Polyak

November 2010.

Boris T. Polyak
1935-2023

Introduction to optimization

Boris T. Polyak

Optimization Software, Inc., 1987

Part 2. Convergence without Optimal Value

- The Second Gradient Descent Lemma

- Convergent Step Size

- Convergence without Optimal Value

Step Size without Optimal Value

- Note that Polyak step size requires the optimal value $f^\star$

Polyak step size: $$\eta_t = \begin{cases} \frac{f(\mathbf{x}_t) - f^\star}{\|\nabla f(\mathbf{x}_t)\|^2}, & \nabla f(\mathbf{x}_t) \neq \mathbf{0} \\ 1, & \nabla f(\mathbf{x}_t) = \mathbf{0} \end{cases}$$

assume known $f^\star$ for a moment

From now on, we try to design step sizes ***without*** the optimal value $f^\star$.

The Second Gradient Descent Lemma

- A second version of gradient descent lemma.

Lemma 2. *Under the same assumptions as Theorem 1. Let* $\{\mathbf{x}_t\}_{t=1}^T$ *be the sequence generated by GD. Then we have*

$$\sum_{t=1}^{T} \eta_t (f(\mathbf{x}_t) - f^\star) \le \frac{1}{2}\|\mathbf{x}_1 - \mathbf{x}^\star\|^2 + \frac{1}{2}\sum_{t=1}^{T} \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2.$$

Proof: The statement can be derived directly from the gradient descent lemma:

$$\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \le \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - 2\eta_t (f(\mathbf{x}_t) - f^\star) + \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2$$

$$\Longrightarrow \eta_t (f(\mathbf{x}_t) - f^\star) \le \frac{1}{2}\left(\|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2\right) + \frac{1}{2}\eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2 \qquad \square$$

Convergence Result

- GD lemma implies the following convergence result.

Lemma 3. *Under the same assumptions as Theorem 1. Let* $\{\mathbf{x}_t\}_{t=1}^T$ *be the sequence generated by GD. Define*

- *best iterate:* $\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$;
- *average iterate:* $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$.

Then, both iterates guarantee that

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \eta_t} + \frac{\sum_{t=1}^T \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^T \eta_t}.$$

Convergence Result

Proof:

- **Case 1:** best iterate $\bar{\mathbf{x}}_T = \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$.

$$\sum_{t=1}^{T} \eta_t (f(\mathbf{x}_t) - f^\star) \geq \left(\sum_{t=1}^{T} \eta_t\right) (f(\bar{\mathbf{x}}_T) - f^\star). \qquad (f(\bar{\mathbf{x}}_T) = \min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t) \leq f(\mathbf{x}_t))$$

Combining the above inequality with Lemma 2 (as restated below),

$$\sum_{t=1}^{T} \eta_t (f(\mathbf{x}_t) - f^\star) \leq \frac{1}{2}\|\mathbf{x}_1 - \mathbf{x}^\star\|^2 + \frac{1}{2}\sum_{t=1}^{T} \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2,$$

we have completed the proof of the desired result:

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^{T} \eta_t} + \frac{\sum_{t=1}^{T} \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^{T} \eta_t}.$$

Convergence Result

Proof:

- **Case 2:** average iterate $\bar{\mathbf{x}}_T = \sum_{t=1}^{T} \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^{T} \eta_t}$.

$$\begin{aligned}
\sum_{t=1}^{T} \eta_t (f(\mathbf{x}_t) - f^\star) &= \left(\sum_{t=1}^{T} \eta_t\right) \left(\sum_{t=1}^{T} \overbrace{\boxed{\frac{\eta_t}{\sum_{t=1}^{T} \eta_t}}}^{\text{(distribution)}} f(\mathbf{x}_t) - f^\star\right) \\
&\geq \left(\sum_{t=1}^{T} \eta_t\right) \left(f\left(\sum_{t=1}^{T} \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^{T} \eta_t}\right) - f^\star\right)
\end{aligned}$$

(Jensen's inequality)

Thus, we achieve the desired result:

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^{T} \eta_t} + \frac{\sum_{t=1}^{T} \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^{T} \eta_t}. \qquad \square$$

Convergent Step Size

Theorem 3. *Under the assumption that $\|\nabla f(\cdot)\| \leq G$. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by the gradient descent method (note that the step size setting cannot use knowledge of T ahead of time). If*

$$\frac{\sum_{t=1}^T \eta_t^2}{\sum_{t=1}^T \eta_t} \to 0 \text{ as } T \to \infty,$$

then we have $f(\bar{\mathbf{x}}_T) \to f^\star$ as $T \to \infty$ for both best and average iterates.

Proof: From the second gradient descent lemma, we know that

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \eta_t} + \frac{\sum_{t=1}^T \eta_t^2 \boxed{\|\nabla f(\mathbf{x}_t)\|^2} \leq G^2}{2\sum_{t=1}^T \eta_t}$$

The condition $\frac{\sum_{t=1}^T \eta_t^2}{\sum_{t=1}^T \eta_t} \to 0$ implies the convergence of the second term.

Moreover, this condition implies $\sum_{t=1}^T \eta_t \to \infty$ (think why?). $\square$

Convergent Step Size

Theorem 3. *Under the assumption that $\|\nabla f(\cdot)\| \le G$. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by the gradient descent method (note that the step size setting cannot use knowledge of T ahead of time). If*

$$\frac{\sum_{t=1}^T \eta_t^2}{\sum_{t=1}^T \eta_t} \to 0 \text{ as } T \to \infty,$$

then we have $f(\bar{\mathbf{x}}_T) \to f^\star$ as $T \to \infty$ for both best and average iterates.

Example:

a typical ***time-varying*** (in fact, decreasing) step sizes:

$$\eta_t = \frac{1}{\sqrt{t}} \Rightarrow \frac{\sum_{t=1}^T \eta_t^2}{\sum_{t=1}^T \eta_t} \approx \frac{\log T}{\sqrt{T}} \to 0.$$

Convergence without Optimal Value

Theorem 4. *Under the same assumptions with Theorem 3. Let* $\{\mathbf{x}_t\}_{t=1}^T$ *be the sequence generated by GD with step size*

$$\eta_t = \frac{1}{\|\nabla f(\mathbf{x}_t)\|\sqrt{t}}.$$

Then

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{G\left(\|\mathbf{x}_1 - \mathbf{x}^\star\|^2 + \log T + 1\right)}{2\sqrt{T}} = \mathcal{O}\left(\frac{\log T}{\sqrt{T}}\right),$$

where $\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ *or* $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$.

This tuning does not require the knowledge of $f^\star$, but the convergence rate exhibits a $\log T$ factor gap.

Convergence without Optimal Value

Proof:

$$
\begin{aligned}
f(\bar{\mathbf{x}}_T) - f^\star &\leq \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \eta_t} + \frac{\sum_{t=1}^T \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^T \eta_t} && \text{(Lemma 3)} \\
&\leq \frac{G\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \eta_t \|\nabla f(\mathbf{x}_t)\|} + \frac{G\sum_{t=1}^T \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^T \eta_t \|\nabla f(\mathbf{x}_t)\|} && (\|\nabla f(\cdot)\| \leq G) \\
&\leq \frac{G\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \frac{1}{\sqrt{t}}} + \frac{G\sum_{t=1}^T \frac{1}{t}}{2\sum_{t=1}^T \frac{1}{\sqrt{t}}} && \begin{array}{l}(\sum_{t=1}^T \frac{1}{t} \leq \log T + 1) \\ (\sqrt{T} \leq \sum_{t=1}^T \frac{1}{\sqrt{t}} \leq 2\sqrt{T})\end{array}
\end{aligned}
$$

Thus,

$$
f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{G\left(\|\mathbf{x}_1 - \mathbf{x}^\star\|^2 + \log T + 1\right)}{2\sqrt{T}} = \mathcal{O}\left(\frac{\log T}{\sqrt{T}}\right). \qquad \square
$$

Part 3. Optimal Rates

- Optimal Result with Known T

- Optimal Result with Unknown T

Towards Optimal Resolutions

Theorem 4. *Under the same assumptions with Theorem 1. Let* $\{\mathbf{x}_t\}_{t=1}^T$ *be the sequence generated by GD with step size*

$$\eta_t = \frac{1}{\|\nabla f(\mathbf{x}_t)\|\sqrt{t}}.$$

Then

$$f(\bar{\mathbf{x}}_T) - f^\star \le \frac{G\left(\|\mathbf{x}_1 - \mathbf{x}^\star\|^2 + \log T + 1\right)}{2\sqrt{T}} = \mathcal{O}\left(\frac{\log T}{\sqrt{T}}\right),$$

where $\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ *or* $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$.

Theorem 2. *Under the same assumptions with Theorem 1. Let* $\{\mathbf{x}_t\}_{t=1}^T$ *be the sequence generated by the gradient descent method with Polyak step size and* $f^\star$ *be the optimal value. Define* $\bar{\mathbf{x}}_T = \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$*, we have*

$$f(\bar{\mathbf{x}}_T) - f^\star \le \frac{G\|\mathbf{x}_1 - \mathbf{x}^\star\|}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right).$$

with Polyak's step size (known $f^\star$***)***

Now, we will improve this to optimality with an additional ***bounded domain*** assumption.

i.e., $\|\mathbf{x} - \mathbf{y}\| \le D$ for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$

Optimal Result with Known T

Theorem 5. *Under the same assumptions with Theorem 1, assume the feasible domain $\mathcal{X}$ is bounded and convex with a diameter $D > 0$, that is, $\|\mathbf{x} - \mathbf{y}\|_2 \leq D$ holds for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by GD with step size*

$$\eta_t = \frac{D}{G\sqrt{T}}.$$

Then

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{DG}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right),$$

where $\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ or $\bar{\mathbf{x}}_T \triangleq \frac{1}{T}\sum_{t=1}^T \mathbf{x}_t$.

Optimal Result with Known T

step size $\eta_t = \frac{D}{G\sqrt{T}}$ $\Longrightarrow$ $f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{DG}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$

$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ or $\bar{\mathbf{x}}_T \triangleq \frac{1}{T}\sum_{t=1}^T \mathbf{x}_t$

Proof: Plugging $\eta_t = \frac{D}{G\sqrt{T}}$ into

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \eta_t} + \frac{\sum_{t=1}^T \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^T \eta_t}$$

$(\|\mathbf{x}_1 - \mathbf{x}^\star\| \leq D)$

$(\|\nabla f(\cdot)\| \leq G)$

Notice that $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t} = \frac{1}{T}\sum_{t=1}^T \mathbf{x}_t.$ $\square$

Optimal Result with Known T

step size $\eta_t = \frac{D}{G\sqrt{T}}$ $\Longrightarrow$ $$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{DG}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$$

$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ or $\bar{\mathbf{x}}_T \triangleq \frac{1}{T}\sum_{t=1}^T \mathbf{x}_t$

- This convergence rate is equivalent to the sample complexity bound $T = \frac{D^2G^2}{\varepsilon^2} = \mathcal{O}(\frac{1}{\varepsilon^2})$ to achieve $f(\bar{\mathbf{x}}_T) - f^\star \leq \varepsilon$.

- $\frac{DG}{\sqrt{T}}$ is already minimax optimal for convex and Lispchitz functions.

- This result needs to know the total round number T in advance.

The last characteristics could be undesirable in practice.

Optimal Result with Unknown T

Theorem 6. *Under the same assumptions with Theorem 1, assume the feasible domain $\mathcal{X}$ is bounded and convex with a diameter $D > 0$, that is, $\|\mathbf{x} - \mathbf{y}\|_2 \leq D$ holds for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by GD with step size*

$$\eta_t = \frac{D}{G\sqrt{t}}. \qquad \text{“anytime” algorithm}$$

Then

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{DG}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right),$$

where $\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=\lceil T/2 \rceil}^T} f(\mathbf{x}_t)$ *or* $\bar{\mathbf{x}}_T \triangleq \sum_{t=\lceil T/2 \rceil}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=\lceil T/2 \rceil}^T \eta_t}$.

Intuition: bounded domain assumption ensures $\|\mathbf{x}_t - \mathbf{x}^\star\|$ (not just $\|\mathbf{x}_1 - \mathbf{x}^\star\|$) to be bounded so that we can avoid the $\mathcal{O}(\log T)$ factor in the analysis.

Optimal Result with Unknown T

Proof: It is easy to extend the second GD lemma from $t = 1, \ldots, T$ to $t = \lceil \frac{T}{2} \rceil, \ldots, T$:

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \eta_t} + \frac{\sum_{t=1}^T \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^T \eta_t}$$

$$\Longrightarrow f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{\|\mathbf{x}_{\lceil \frac{T}{2} \rceil} - \mathbf{x}^\star\|^2}{2\left(\sum_{t=\lceil \frac{T}{2} \rceil}^T \eta_t\right)} + \frac{G^2}{2} \frac{\sum_{t=\lceil \frac{T}{2} \rceil}^T \eta_t^2}{\sum_{t=\lceil \frac{T}{2} \rceil}^T \eta_t}$$

$$\left(\sum_{t=\lceil \frac{T}{2} \rceil}^T \frac{1}{\sqrt{t}} \geq \frac{T}{2} \cdot \frac{1}{\sqrt{T}} = \frac{\sqrt{T}}{2}\right) \qquad \leq \frac{DG}{2} \frac{1}{\boxed{\sum_{t=\lceil \frac{T}{2} \rceil}^T \frac{1}{\sqrt{t}}}} + \frac{DG}{2} \boxed{\frac{\sum_{t=\lceil \frac{T}{2} \rceil}^T \frac{1}{t}}{\sum_{t=\lceil \frac{T}{2} \rceil}^T \frac{1}{\sqrt{t}}}} \approx \sqrt{T}$$

$$\left(\sum_{\lceil T/2 \rceil}^T \frac{1}{t} \leq \log(T+1) - \log(\lceil T/2 \rceil) \leq \log(3)\right)$$

$$\Longrightarrow f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{DG}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right). \qquad \square$$

"Parameter-Free" Extension

- Previous discussions focus on the ***bounded*** domain.
- Step size tuning in ***unbounded*** domain is always nontrivial.

Algorithm 1 DoG with SGD [Ivgi et al., 2023]

Input: feasible domain $\mathcal{X}$ (which can be unbounded); initial point $\widehat{\mathbf{x}}_0 \in \mathcal{X}$; step size $\{\eta_t\}_{t=1}^T$; a small constant $r_\varepsilon > 0$.

1: Set $\eta_0 = \frac{r_\varepsilon}{\|\mathbf{g}_0\|}$

2: **for** $t = 1$ **to** $\cdots$ (maybe T) **do**

3: Perform the SGD update

$$\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}[\mathbf{x}_t - \eta_t \mathbf{g}_t], \tag{4}$$

where $\mathbf{g}_t$ is the stochastic gradient of f at $\mathbf{x}_t$ and the step size is set as

$$\eta_t = \frac{\bar{r}_t}{\sqrt{\sum_{s=1}^t \|\mathbf{g}_s\|^2}}, \text{ where } \bar{r}_t \triangleq \max_{s\in[t]} \max\{\|\mathbf{x}_s - \mathbf{x}_0\|, r_\varepsilon\} \tag{5}$$

4: **end for**

Output: weighted average $\bar{\mathbf{x}}_t = \frac{1}{\sum_{s=0}^{t-1} \bar{r}_s} \cdot \sum_{s=0}^{t-1} \bar{r}_s \mathbf{x}_s$.

Ivgi, Maor, Oliver Hinder, and Yair Carmon. DoG is SGD's Best Friend: A Parameter-Free Dynamic Step Size Schedule. ICML 2023.

“Parameter-Free” Extension

Assumption 1 (convexity)**.** The function $f : \mathcal{X} \mapsto \mathbb{R}$ is convex.

Assumption 2 (domain boundedness)**.** The feasible domain $\mathcal{X}$ is convex and bounded by D, that is, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, we have $\|\mathbf{x} - \mathbf{y}\| \le D$.

Assumption 3 (boundedness of gradient estimates)**.** The norm of gradient estimates is bounded by G, that is, for any $\mathbf{x} \in \mathcal{X}$, we have $\|\widetilde{\nabla} f(\mathbf{x})\|_* \le G$.

Theorem 1. *Under Assumptions 1–3, the* DoG *algorithm (Algorithm 1) achieves the following convergence guarantee:*

$$\mathbb{E}\left[f(\bar{\mathbf{x}}_t) - f_*\right] \le \mathcal{O}\left(\frac{DG}{\sqrt{T}} \log_+\left(\frac{D}{r_\varepsilon}\right)\right), \tag{6}$$

where D and G are the upper bounds of the domain diameter and the stochastic gradient norm, as defined in Assumptions 2 and 3, respectively. Notably, those constants (D and G) are not required as the algorithmic input.

Results so far

- For convex and G-Lipschitz functions,
  GD with different settings (step size, output sequence) achieve

	Step Size	Output Sequence	Convergence Rate	Remark
(1)	$\eta_t = \frac{f(\mathbf{x}_t)-f^\star}{\\|\nabla f(\mathbf{x}_t)\\|^2}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$	$\mathcal{O}(1/\sqrt{T})$	optimal Polyak's step size require $f^\star$
(2)	$\eta_t = \frac{1}{\\|\nabla f(\mathbf{x}_t)\\|\sqrt{t}}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$	$\mathcal{O}(\log T/\sqrt{T})$	suboptimal
(3)	$\eta_t = \frac{D}{G\sqrt{T}}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$	$\mathcal{O}(1/\sqrt{T})$	bounded domain require T
(4)	$\eta_t = \frac{D}{G\sqrt{t}}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=\lceil T/2\rceil}^T} f(\mathbf{x}_t)$ $\bar{\mathbf{x}}_T \triangleq \sum_{t=\lceil T/2\rceil}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=\lceil T/2\rceil}^T \eta_t}$	$\mathcal{O}(1/\sqrt{T})$	bounded domain

Revisit the Value of Polyak Step Size

For *unconstrained* optimization, Polyak step size is very simple and enjoys many good properties (e.g., "universality"); though estimating $f^\star$ is a problem to explore.

In this landscape of stepsize selection strategies, the Polyak stepsize, proposed by Polyak [1969] stands out for its theoretical elegance and convergence properties. Starting from an arbitrary $\boldsymbol{x}_1 \in \mathbb{R}^d$, the Polyak stepsize is defined as

$$\boldsymbol{x}_{t+1} = \boldsymbol{x}_t - \frac{f(\boldsymbol{x}_t) - f^\star}{\|\boldsymbol{g}_t\|^2}\boldsymbol{g}_t, \tag{1}$$

where $\mathbf{0} \neq \boldsymbol{g}_t \in \partial f(\boldsymbol{x}_t)$ and $f^\star = \min_{\boldsymbol{x}} f(\boldsymbol{x})$. If $\boldsymbol{g}_t = \mathbf{0}$, then $\boldsymbol{x}_{t+1} = \boldsymbol{x}_t$. This update rule can achieve linear convergence for strongly convex and smooth functions, $\mathcal{O}(1/T)$ rate for convex smooth functions, and $\mathcal{O}(1/\sqrt{T})$ rate for non-smooth convex ones. This is particularly interesting because all of these rates are achieved with a unique stepsize and without knowledge of smoothness or curvature constants. In other words, this update rule is *adaptive* to the geometry of the functions to optimize.

F. Orabona, R. D'Orazio. New Perspectives on the Polyak Stepsize: Surrogate Functions and Negative Results. NeurIPS 2025.

Part 4. Strongly Convex and Lipschitz

- Strong Convexity

- Convergence Result

Strongly Convex and Lipschitz

Theorem 7. *Under the same assumptions with Theorem 1, except that f is σ-strongly-convex. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by GD with step size*

$$\eta_t = \frac{2}{\sigma(t+1)}.$$

Then (i)

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{2G^2}{\sigma(T+1)} = \mathcal{O}\left(\frac{1}{T}\right),$$

where $\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ *or* $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{2t}{T(T+1)} \mathbf{x}_t$.
And (ii)

$$\|\bar{\mathbf{x}}_T - \mathbf{x}^\star\| \leq \frac{2G}{\sigma\sqrt{T+1}}.$$

Strongly Convex and Lipschitz

Proof: we start by extending ***the first GD lemma*** to strongly convex case.

Strongly convex case:

$$
\begin{aligned}
\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 &\le \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - 2\eta_t \langle \nabla f(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}^\star \rangle + \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2 \\
&\le \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - 2\eta_t \left(f(\mathbf{x}_t) - f^\star + \frac{\sigma}{2}\|\mathbf{x}_t - \mathbf{x}^\star\|^2 \right) + \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2
\end{aligned}
$$

(strong convexity: $f(\mathbf{x}_t) - f(\mathbf{x}^\star) + \frac{\sigma}{2}\|\mathbf{x}_t - \mathbf{x}^\star\|^2 \le \langle \nabla f(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}^\star \rangle$)

$$
\le (1 - \sigma\eta_t) \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - 2\eta_t \left(f(\mathbf{x}_t) - f^\star \right) + \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2
$$

$$
\Longrightarrow f(\mathbf{x}_t) - f^\star \le \frac{\eta_t^{-1} - \sigma}{2}\|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \frac{\eta_t^{-1}}{2}\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 + \frac{\eta_t G^2}{2}
$$

(rearranging)

Strongly Convex and Lipschitz

$$f(\mathbf{x}_t) - f^\star \le \frac{\eta_t^{-1} - \sigma}{2}\|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \frac{\eta_t^{-1}}{2}\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 + \frac{\eta_t G^2}{2}$$

$$= \frac{\sigma}{4}\left((t-1)\|\mathbf{x}_t - \mathbf{x}^\star\|^2 - (t+1)\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2\right) + \frac{G^2}{\sigma(t+1)}$$

$$\Longrightarrow \quad t(f(\mathbf{x}_t) - f^\star) \le \frac{\sigma}{4}\left((t-1)t\|\mathbf{x}_t - \mathbf{x}^\star\|^2 - t(t+1)\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2\right) + \frac{G^2}{\sigma}$$

telescope now

$$\Longrightarrow \quad \sum_{t=1}^{T} t(f(\mathbf{x}_t) - f^\star) \le \frac{\sigma}{4}\left(0 \cdot 1 \cdot \|\mathbf{x}_1 - \mathbf{x}^\star\|^2 - T(T+1)\|\mathbf{x}_{T+1} - \mathbf{x}^\star\|^2\right) + \frac{G^2 T}{\sigma} \le \frac{G^2 T}{\sigma}$$

Next step: relating $\sum_{t=1}^{T} t(f(\mathbf{x}_t) - f(\mathbf{x}^\star))$ to $f(\bar{\mathbf{x}}_T) - f(\mathbf{x}^\star)$.

Strongly Convex and Lipschitz

Recall that the output sequence is $\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ or $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{2t}{T(T+1)} \mathbf{x}_t$.

Case 1: $$\sum_{t=1}^T t(f(\mathbf{x}_t) - f^\star) \geq \left(\sum_{t=1}^T t\right)(f(\bar{\mathbf{x}}_T) - f^\star) = \frac{T(T+1)}{2}(f(\bar{\mathbf{x}}_T) - f^\star)$$

Case 2: $$\sum_{t=1}^T t(f(\mathbf{x}_t) - f^\star) = \sum_{t=1}^T t f(\mathbf{x}_t) - \frac{T(T+1)}{2} f^\star = \frac{T(T+1)}{2}\left(\sum_{t=1}^T \underbrace{\boxed{\frac{2t}{T(T+1)}}}_{\text{(distribution)}} f(\mathbf{x}_t) - f^\star\right)$$

$$\underbrace{\geq}_{\text{(Jensen's inequality)}} \frac{T(T+1)}{2}(f(\bar{\mathbf{x}}_T) - f^\star)$$

(*i*) is proved. $\square$

Strongly Convex and Lipschitz

Proof: *(ii)* can be derived directly from *(i)* and strong convexity.

$$\frac{\sigma}{2}\|\bar{\mathbf{x}}_T - \mathbf{x}^\star\|^2 \leq \langle \nabla f(\mathbf{x}^\star), \bar{\mathbf{x}}_T - \mathbf{x}^\star \rangle + \frac{\sigma}{2}\|\bar{\mathbf{x}}_T - \mathbf{x}^\star\|^2 \underset{\text{(strong convexity)}}{\leq} f(\bar{\mathbf{x}}_T) - f^\star \underset{(i)}{\leq} \frac{2G^2}{\sigma(T+1)}$$

(first-order optimality condition: $\langle \nabla f(\mathbf{x}^\star), \mathbf{x} - \mathbf{x}^\star \rangle \geq 0$)

Thus, we prove that no matter for which constructions of $\bar{\mathbf{x}}_T$, it holds that

$$\|\bar{\mathbf{x}}_T - \mathbf{x}^\star\| \leq \frac{2G}{\sigma\sqrt{T+1}}.$$

(*ii*) is proved. $\square$

Summary

Table 1: A summary of convergence rates of GD method.

Function Family	Step Size	Output Sequence	Convergence Rate	Remark
convex and G-Lipschitz	$\eta_t = \frac{f(\mathbf{x}_t)-f^\star}{\\|\nabla f(\mathbf{x}_t)\\|^2}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$	$\mathcal{O}(1/\sqrt{T})$	optimal Polyak's step size require $f^\star$
	$\eta_t = \frac{1}{\\|\nabla f(\mathbf{x}_t)\\|\sqrt{t}}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$	$\mathcal{O}(\log T/\sqrt{T})$	suboptimal
	$\eta_t = \frac{D}{G\sqrt{T}}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$	$\mathcal{O}(1/\sqrt{T})$	bounded domain require T
	$\eta_t = \frac{D}{G\sqrt{t}}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=\lceil T/2\rceil}^T} f(\mathbf{x}_t)$ $\bar{\mathbf{x}}_T \triangleq \sum_{t=\lceil T/2\rceil}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=\lceil T/2\rceil}^T \eta_t}$	$\mathcal{O}(1/\sqrt{T})$	bounded domain
σ-strongly convex and G-Lipschitz	$\eta_t = \frac{2}{\sigma(t+1)}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$	$\mathcal{O}(1/T)$	$\\|\bar{\mathbf{x}}_T - \mathbf{x}^\star\\|$ is bounded

Summary

GRADIENT DESCENT
- Convex Optimization Problem
- Gradient Descent
- The First Gradient Descent Lemma

OPTIMAL IN CONVEX AND LIPSCHITZ CASE
- Optimal Result with Known T
- Optimal Result with Unknown T

POLYAK'S STEP SIZE
- Polyak's Step Size
- Convergence
- Convergence Rate

STRONGLY CONVEX AND LIPSCHITZ
- Strong Convexity
- Convergence Result

CONVERGENCE WITHOUT OPTIMAL VALUE
- The Second Gradient Descent Lemma
- Convergent Step Size
- Convergence without Optimal Value

Q & A

Thanks!